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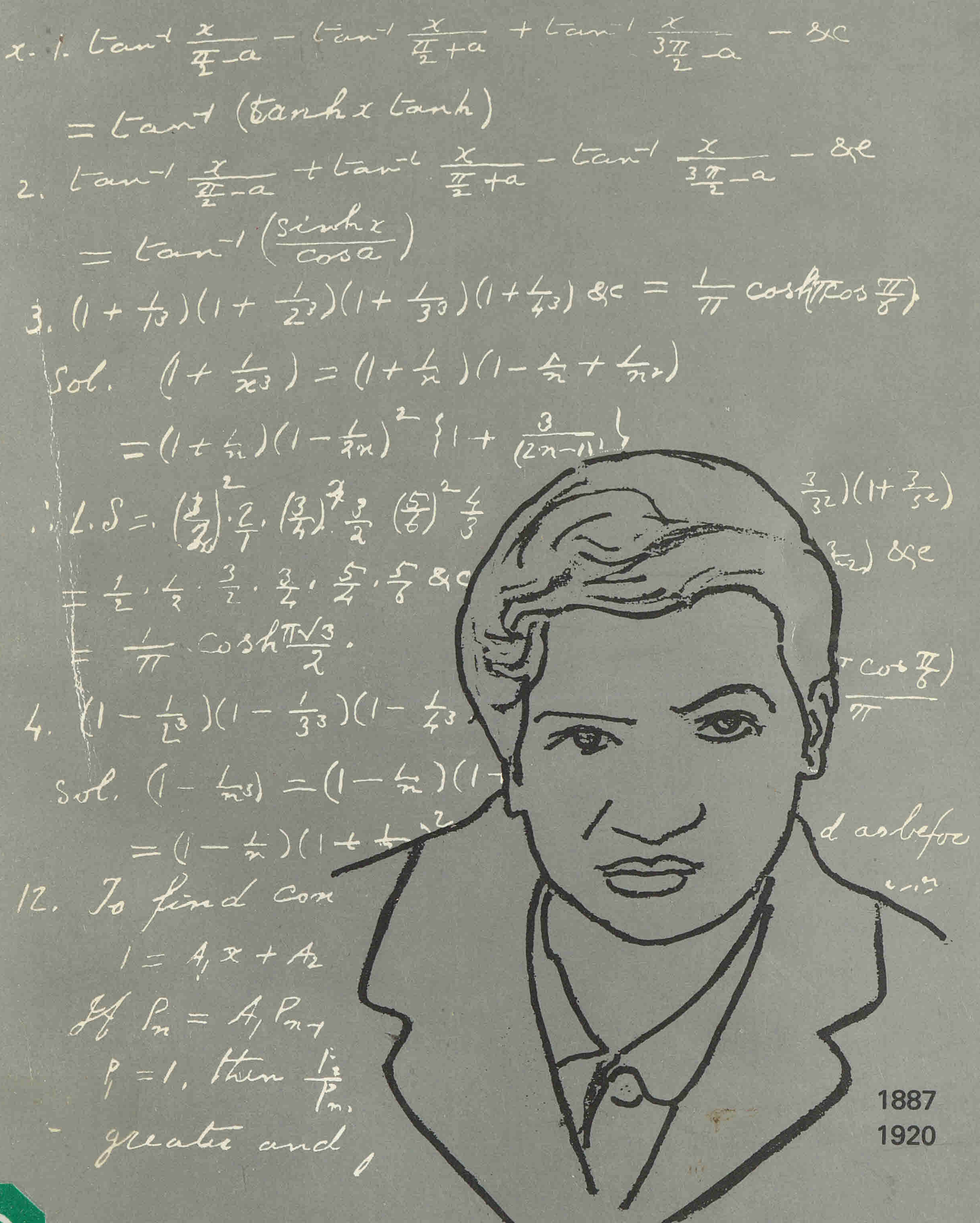

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SPECIAL ISSUE OF THE JOURNAL OF THE
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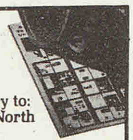

Special Issue

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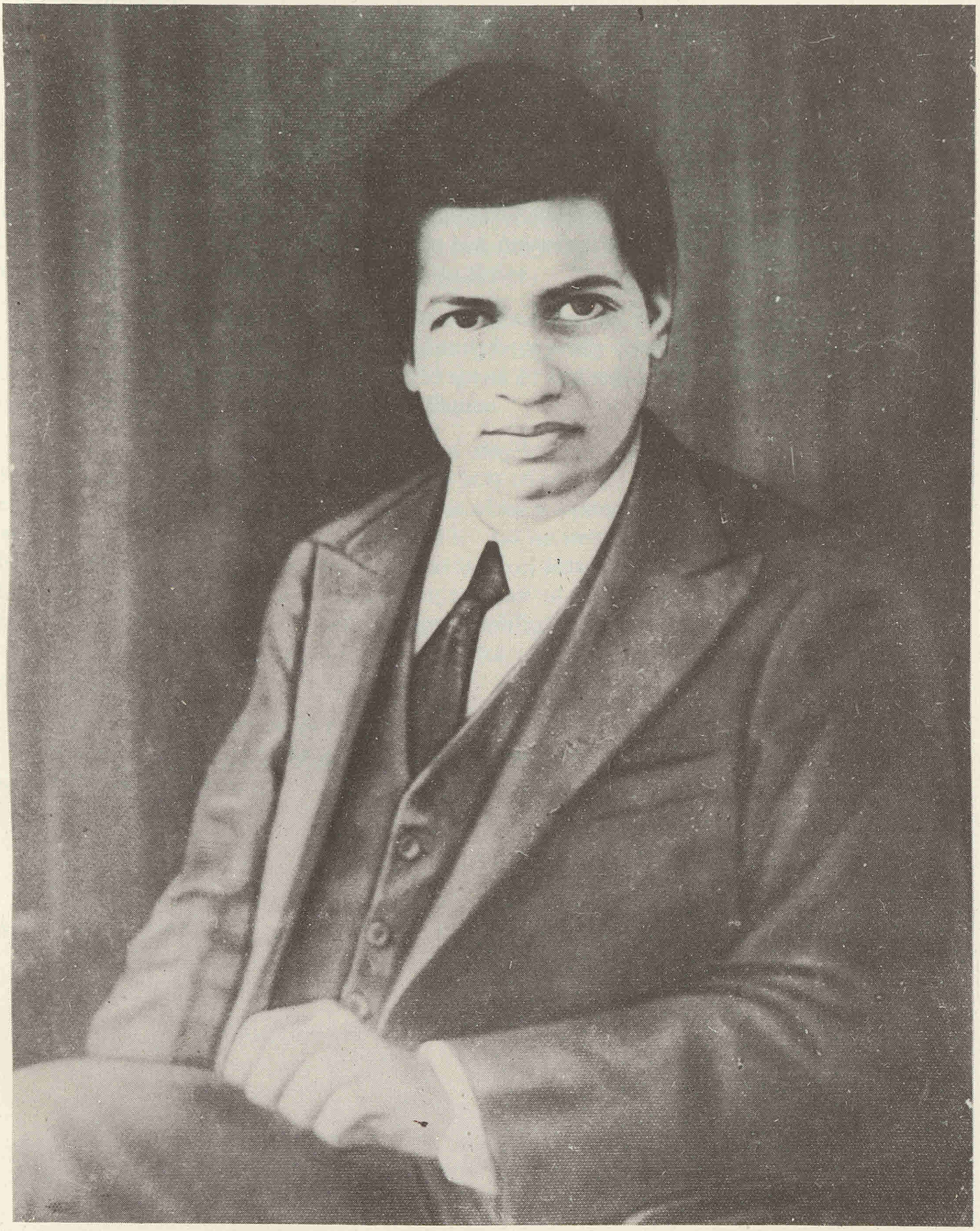

Srinivasa Ramanujan
December 22, 1887 – April 26, 1920

FOREWORD

The years 1987, 88 and 89 are of particular significance to us in this country. They are the Birth Centenaries of Srinivasa Ramanujan, Chandrasekhara Venkata Raman, and Jawaharlal Nehru. Each of these individuals reached extraordinary heights of achievement in their spheres of activity – Mathematics, Natural Science and Statesmanship. Their lives and accomplishments are precious reminders and inspiration to us – in these days when it seems more fashionable to run ourselves down – of what we can achieve with talent, perseverance and human qualities.

Like Abel and Galois before him, Ramanujan too had only a brief life. Of this, less than 7 years were spent in a place and atmosphere really conducive to creative work. He had no formal training in Mathematics to speak of and was truly an untutored genius, an uncut diamond but of uncommon brilliance.

We at this Institute have arranged a day-long symposium to honour Srinivasa Ramanujan on his Birth Centenary, as our way of paying tribute to someone whose only equals are Euler and Gauss. In the presence of such a phenomenon, one can only wonder what forces and traditions could have led to this occurrence. It is our good fortune that he was one of us. It is unfortunate that too little of Ramanujan's life and work – esoteric though the latter is – seem to be known to most of us. We have tried to gather some eminent mathematicians who are the best qualified to speak of Ramanujan, to address us on this occasion. We are sincerely grateful to all of them for having accepted our invitation.

This booklet has been prepared as a special issue of the Journal of the Indian Institute of Science, so that the talks given at this symposium may reach a wider audience of students and teachers.

September, 1987

C N R RAO
Director

SRINIVASA RAMANUJAN CENTENARY 1987

A Special issue of the Journal of the Indian Institute of Science

CONTENTS

By way of an introduction . . .

Srinivasa Ramanujan was a truly remarkable mathematician, one who towered over the accumulated mathematical wisdom of the world of his time. The legacy he has left is so rich and fascinating that his contributions would continue to occupy the most creative of mathematicians for a long time.

Ramanujan spent his early years indulging in the joy of (re)discovering many classical results in infinite series, definite integrals, Bernoulli numbers and gamma function. He had found out for himself a form of the prime number theorem, sums of arithmetical functions and a beautiful result on the abundance of composite numbers with about log log n prime factors. He filled three notebooks with a plethora of results, more in the nature of a personal memoir, than a mathematical monograph. Most of what he did later appears to have roots in these results.

He made profound contributions to the theory of partitions: congruence properties, asymptotic properties, and some fabulous identities connecting infinite continued fractions and infinite products. Of two related theorems, which Ramanujan stated in his (now famous) letter to Hardy in 1913, Hardy's comments were, "... a single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them". Ramanujan, in collaboration with Hardy, derived the remarkable formula for p(n), and his approach to the evaluation of p(n) has beome a standard tool in additive number theory.

His results on definite integrals, some connecting hypergeometric functions and the gamma function are objects of sheer beauty. His work on elliptic functions, modular equations, in particular those with singular moduli, constitutes an independent and original theory equalling those of Euler, Gauss, Jacobi and Heine.

His work on the representation of integers as sums of squares, led him to discover elementary proofs of identities related to Jacobi's elliptic theta function. The beautiful theories of modular functions, Dirichlet series with Euler products, a summation named after him, and the arithmetical function $\tau(n)$ have contributed to many recent and profound developments in arithmetic algebraic geometry.

He continued to carry out in his deathbed prolific work on hypergeometric series, to which he had devoted two chapters in his notebooks. Some of these last results, found recently in his 'Lost' Notebook, contain many elegant summations which are profound generalizations of results discovered by Euler, Gauss and Jacobi.

Ramanujan's works have made a profound impact in some areas of modern number theoretic research. One sees also recent applications of his results, based on asymptotic methods in combinatory analysis to the analysis of complexity of algorithms in computer science, on partition identities to some problems in statistical mechanics and in some beautiful generalizations based on bijective proofs in partition theory.

Hardy judged Ramanujan's work as, "it has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange.. One gift it has which no one can deny, profound and invincible originality". While the second statement is undisputably accepted as true, the first statement has come under criticism in the light of the many results discovered by Ramanujan in the last year of his life, and which are better understood today. According to Askey very few of these results have been discovered in the more than half a century between when Ramanujan discovered them and when his pages were found. These go further to project Ramanujan as an extraordinary mathematician.

C. E. VENI MADHAVAN

J. Indian Inst. Sci. pp. 3–9
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The Indian tradition in mathematics

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We are quite aware of centres of excellence in science in the western hemisphere. To list a few names: Oxford, Cambridge, Harvard, the Massachusets Institute of Technology, The Institute for Advanced Study in Princeton. The fame of these centres could be traced to renowned scientists who were associated with these centres and created landmarks in science in their investigations. To cite just one instance, it cannot be denied that the association of Sir Isaac Newton with Cambridge did make Cambridge better known. Norbert Wiener, more recently, made the MIT more famous. It is also true that these scientists more often left behind them illustrious followers to continue their work in one or more disciplines. A tradition for work in a branch of science was thus established in such centres. A Hilbert in Göttingen, a Poincaré in Paris and so on make one ascribe a tradition for the whole of their native land, Germany or France as the case may be. All these pertain to rather modern times. Any tradition which goes farther back to medieval or ancient times points out to an older civilization with a bent on scientific investigation. People speak about the Greek tradition in geometry continuing to exist with present day Italian geometers. There are western students of history of science who admit of Babylonian tradition which, they claim, was the forerunner for any Indian contribution or even the source for it. In this context, the question arises as to whether we had a tradition in mathematics, whether it continued for long, whether it died out and if so when and why. If there had been a tradition we should also know what its chief contributions were. The purpose of this talk is to point out the existence of a tradition and to provide satisfactory answers to the issues raised just above.

Before we begin, a word about dating source material is in order. Before the advent of the westerner into the Indian soil our forefathers were indifferent to dating of any work. In most of the cases they were indifferent to the authorship too. The reason was either a philosophical attitude that the facts were just discovered by God's will or that the utility of the fact was more important. The dating suggested by western scholars for many of the works of the east is at best a conjecture. Be it by archaeological procedures or by philological ones, the norms used by the western scholar applies to their region at the pace of evolution in the recent past. In the east we had standardization thousands of years back and consequently the need for change was not much so that the effect of evolution is not so well discernible. Anyway, for convenience of understanding we shall stick to the dates assigned by western scholars. There is scope to think that our ṛshis were scientists, their yāgas were experiments or applications and their vedis laboratories, but we shall deal here only with more specific things pertaining to mathematics.

The oldest work extant in India are the Vedas. The Vedas have been preserved by a word of mouth tradition, the best non-perishable record unless and until the population knowing them is completely annihilated. Other modes of preservation and propogation of knowledge are of more recent origin. The Vedas, even today, give scope for newer and newer interpretations. They can be deemed as a compilation of scientific findings or ideas or of historical events, or, as is more often considered, of philosophical truth. The westerner dates these scriptures as pertaining to 5000 B.C., most probably because he does not believe the existence of the human race before Adam and Eve. Anyway he admits that these scriptures of the Hindus are amongst the oldest available literature of the world. The Vedas have introduced the decimal system and we have the powers of 10: daśa, śata, sahasra, ayutha... . Rational numbers were known during the Vedic times: pāda ($\frac{1}{4}$), ardha ($\frac{1}{2}$), tripāda ($\frac{3}{4}$) indicate their thinking about subdividing the unit into equal parts corresponding to the denominator and taking multiples of the subdivision. The idea in the Roman notation for 9 *viz* IX, can be traced to the use of the words like Ekonavimśati (for 19) in the Vedas.

The Ṛgveda indicates that the priests of those times had sufficiently precise knowledge of astronomical elements. More specifically, for instance, the priests of Atri family could predict a solar eclipse*. During the Yajurveda period it was known* that the solar year is more than 364 days but less than 366 days amounting to 365 days and a little more. The above two aspects of knowledge of astronomy, besides revealing keenness of observation of the heavens during the period point to a capability for deeper arithmetical calculations. Most surprisingly, the Indian tradition in mathematics up to the medieval period has been based on astronomical studies. The knowledge revealed by the Vedas should have had its origin much earlier to the Vedas, the latter being a compilation of important information available in their or earlier times. Thus the Vedas themselves point to a tradition which continued in the post-Vedic period as revealed by the work Vedānga Jyotiṣa of Lagadha. Later to this (Vedānga Jyotiṣa), we have the Śulvasutras** (Baudhāyana, Āpasthamba, Kātyayana, Mānava) which are dated between 800 and 500 B.C. Śulva means measurement and the Sutras relate to the construction of Vedis (mounds for performance of yāgas) and agnis (altars for keeping fire). These sūtras indicate a knowledge of what we now call the Pythogoras theorem (580–500 B. C.)*** through a number of relations of the form $a^2 + b^2 = c^2$. The recorded evidence of knowledge of this kind is *not* so marked in the Egyptian papyrus though evidence does exist about Babylonian knowledge of squared relationships in their cuneiform tablet which is dated to pertain to 2000 B.C. The Indian tradition of recording only facts without proofs has led some westerners to conjecture that the Śulvasūtras are post-Pythogorean. The irrationality of $\sqrt{2}$, the constancy of the ratio of the circumference of a circle to its diameter, the irrationality of this constant value as revealed by some approximations to π were known during the Śulvasūtra period. Geometrical construc-

* *see* Vedānga Jyotiṣa (ed. T. S. Kuppanna Sastry).

** (ed.) S. N. Sen and A. K. Bag (Indian National Science Academy, 1983).

*** A formal proof of this theorem is due to Euclid (300 B.C.).

tions such as the construction of a square, the construction of a square equal in area to a given rectangle, the construction of a square equal in perimeter to a circle, etc., are found in Śulvasūtras.

Other ancient cultures in India did also have the mathematical tradition. For instance, the oldest Tamil verse relating to mathematics known to carpenters in Tamil Nadu is as ascribed to one Kākkaippādiniyār who was a poet of the pre-Tholkāppiam period. The content of the verse is that

$$\frac{\text{circumference}}{2} \times \frac{\text{diameter}}{2}$$

gives the area of a circle. As with the Śulvasūtras, knowledge of the Pythogorean theorem can also be claimed for the Tamil tradition. That different units are needed for different situations involving quantification is expressed in a pre-Sangam Vaishnavite work. The Tamils did go in for small enough fractions such as Immi (1/1075200).

It is well-known that there was a decline of the Vedas forced by the advent of Buddhism and Jainism. There is scope to believe that the Jains did maintain the Hindu tradition in mathematics though they did not accept the Vedas. However, no work pertaining to this period seems to be available. Mahavirá's later work (800–1100 A. D.) indicates a possible continuation of the tradition of earlier origin.

With Hindu renaissance the mathematical tradition was also restored. The first famous Indian mathematician-astronomer relating to this period is Āryabhata who is famous because of his work entitled Āryabhatīya (499 A. D.). Āryabhatī ya contains a part, named Ganitabhāga devoted to mathematics. This part points out to a knowledge of the trigonometric ratio of sine (jyā), algorithms to compute the square root and cube root of a number, approximation to π to four decimal places, etc. The interesting fact in the Āryabhatīya is the solution for x, y in integers of the equation

$$ax + by = c$$

where a, b, c are integers, with some 'minimality' for x, y. The procedure adopted (rather, the algorithm, in the modern sense) is through what is known as a finite continued fraction, in present day language, and is much the same as what Euler did to solve this so-called 'linear diophantine equation' in the 17th century. The Āryabhatīya has attracted the attention of a number of later scholars who wrote commentaries on it. Of these commentaries the one by Nīlakanṭa Somayāji of Gārga gotra belonging to Kerala stands out as an elaborate commentary which contains very useful information. We will later see how Nīlakanṭa was one of the torch-bearers of the tradition handed over by Āryabhata which existed as a living one till we succumbed to the western type of education and thinking. Varāhamihira of the sixth century A.D. did have the influence of the Āryabhata school but introduced other schools of thought in astronomy in his Panchasiddhāntika. A few modern scholars think that the Pauliśa and Romaka Siddhānthas included in the Panchasiddhāntika were probably Greek or Greco-Roman. I take this opportunity to remind these scholars about the advice of the late savant T. S. Kuppanna Sastry: Don't conclude that some knowledge was borrowed from a source simply because some astronomical constants are found to be the same in that source. They can't but be the same. Again Romaka or Romaśa is an oft-occurring name in our scriptures of yore. Explicit knowledge of aspects of mathematics is revealed in Brahmagupta's work relating to 598–665 A. D. Introduction of negative numbers, area of a cyclic quadrilateral, a precursor to Stirlingś interpolation formula, solution of indeterminate quadratic equations of the form $x^2 + NY^2 = 1$, are some of the things found in Brahmagupta's work. The Sūryasiddhānta (whose authorship is not known, 650–950 A.D.) embodies a knowledge of spherical trigonometry. It indicates the fact that some astronomical instruments were known to the Indians then. This Siddhānta was either pre- or post-Brahmagupta work. Both Brahmasphutasiddhanta of Brahmagupta and the Sūryasiddhānta were different from the Āryabhatan school of thought. Later to Brahmagupta we have the Jainistic school to which the well-known Mahāvira, the author of Gaṇithasārasangraha (800–1100 A. D.) belongs. The distinction of this work is that it is a purely mathematical work and not an auxiliary to any astronomical treatise. The Jainistic school records a knowledge of permutations and combinations, indices and their laws, rudiments of the logarithm (introduced by Napier only in the 17th century in the west), sequences, countability and uncountability of sets. In particular, the Gaṇithasārasangraha deals with geometric progressions, sums of the first n natural numbers, their squares and their cubes, approximate area of an ellipse, etc.

Next to this period came in Bhāskarācharya II (1114 A. D.), perhaps the medieval Indian mathematician whose name is the most popular even now. That he wrote Līlāvati is known to many who are interested in mathematics. There are stories too prevalent about Bhāskara's mastery of astrology. In fact, Āryabhata, a predecessor to Bhāskara, is now known mainly because the first Indian artificial satellite was named after him. Bhaskara's work is known as Siddhāntasiromani and is divided into four parts–Līlāvati (pertaining to arithmetic). Bījaganitha (algebra), Golādhyaya (spherical trigonometry), Grahaganitha (astronomy). Numerous commentaries, including Bhāskara's own (by name Vāsana), have been written on this work. We notice Āryabhatan influence in this work. Bhāskara seems to have been influenced in his astronomical studies by Brahmagupta too as is seen by his work Karanakutūhala. A complete study of solution of linear diophantine equations and quadratic indeterminate equations, ideas which could be traced to the origins of the calculus (*e.g.* $d(\sin x) = \cos x \, dx$ in modern language) are highlights of Bhaskara's work. Concern for approximations to π and irrational numbers shows that the tradition in mathematics enshrined in Vedic and post-Vedic scriptures continued up to the time of Bhāskara II. Some scholars*, both western and Indian, have unfortunately concluded that no progress was made in Indian mathematics and astronomy after Bhāskara II or that he was the last eminent Indian astronomer. The rest of the present talk is devoted to refute this assertion and to point out that there were scholars in Kerala who maintained the Āryabhatan tradition till the western way of education started to throttle traditional Indian learning. That the Āryabhatan tradition was continued in Kerala and was improved on is discernible from the commentary on Āryabhateeyam or the Aryabhatīyabhāshya of Nīlakanṭa Somayāji about which mention

* *See* G. R. Kaye, *Indian mathematics*, Calcutta, 1915, p. 24; Ārka Somayaji, *A critical study of the ancient Hindu astronomy*, Dharwar, 1971, p. 3.

was made earlier. Nīlakaṇṭa's time is between 1444 and 1545 A.D. In this commentary (*see* Chapter II, verse 10, Trivandrum Sanskrit series, No. 2, Part I) Nīlakaṇṭa says "*Kutah punar vāstavim samkhȳamutsrjya āsannaivehokta iti ekenaiva nīamānayorubayoh kwapi na niravayatvam syāt*"— Why then only an approximate value, not the exact value, is given? Because it cannot be. There will never be commensurability for both (the diameter and circumference) with reference to the same unit of measurement. This is a lucid explanation of the irrationality of π for which a formal proof was given by Lambert in 1761 A.D. Nīlakaṇṭa had his own original work Tantrasangraha, a complete copy of which in grantha palm-leaf script is preserved in the Adyar Library at Madras. Most ironically a British civil servant, C. M. Whish, had access to this work and pointed out in 1835 (see *Trans. Royal Asiatic Soc. Gt Br. Ireland*, Vol. 3, pp. 509–523) that the Hindus had known the ideas attributed to Newton and other mathematicians much earlier. Whish had also referred to the later work of the Kerala mathematicians in his essay on the subject but Tantrasangraha was the earliest work he referred to and its copy is in full agreement with Whish's version inclusive of the order of appearance of the verses.

Let me describe to you the nature of the mathematical assertions contained in Tantrasangraha in modern terminology but with the minimum technical jargen. Consider the sequence of numbers

$$\frac{1}{2}, \frac{1}{2}+\frac{1}{2^2}, \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}, \ldots .$$

Denote the nth entry of this sequence by S_n. Then

$$S_n = \frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}$$

so that

$$\frac{1}{2} S_n = \frac{1}{2^2}+\ldots+\frac{1}{2^n}+\frac{1}{2^{n+1}}.$$

Thus $\frac{1}{2} S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$ as seen by subtraction, which is the same as $S_n = 1 - \frac{1}{2^n}$. It is clear that by making n large enough you can make $\frac{1}{2^n}$ smaller than an assigned positive real number. (In fact, S_n is nearer and nearer one as n increases). This, in modern terminology, means that S_n converges to the limit 1 as $n \rightarrow \infty$. Equivalently, we *write*

$$\frac{1}{2}+\frac{1}{2^2}+\ldots = 1.$$

What was discussed just now corresponds to

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}$$

where $a = \frac{1}{2}$, $r = \frac{1}{2}$. This latter assertion is found in Nīlakaṇṭa's Āryabhatīyabhasya. This idea of convergence to a limit amounting to the difference between the limit and the sequential elements becoming 'infinitesimal' was first propounded by Newton in England, and almost simultanenously, by Wilhelm Leibniz in Germany, in the 17th century. The Tantrasangraha, among other things, enunciates the following

$$\frac{\pi}{4} = 1 - \frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

in the sense that the sequence

$$1,\ 1-\frac{1}{3}=\frac{2}{3},\ 1-\frac{1}{3}+\frac{1}{5}=\frac{13}{15}, \ldots$$

converges to $\frac{1}{4}th$ of π, the constant ratio between the circumference and diameter of a circle. The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is now known as Gregory series after its western author James Gregory (1971 A.D.). There is sufficient evidence to show that this series, as also some others, were known to or due to Mādhava (*C* 1350–1425 A.D.) of Sangamagrāma or modern Irinjālakuda of present-day Kerala. Thus the seeds of modern mathematical analysis were sown in our soil at least 2 to 3 centuries earlier to the ideas of the subject occurring in the western world. Among other series known to the Hindus earlier to the westener are the series for the trigonometric sine and cosine functions. A more remarkable feature about this Hindu work is that it was noticed by the Hindus that the Gregory series converges slowly in the sense that to get a close approximation to $\frac{\pi}{4}$ one has to take an element of very high rank in the sequence

$$1,\ 1 - \tfrac{1}{3},\ 1 - \tfrac{1}{3} + \tfrac{1}{5},\ \ldots$$

They attempted at series which converged faster to $\frac{\pi}{4}$so that an element of a much lower rank would serve the purpose. For instance, one such series is

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3-3} - \frac{1}{5^3-5} = \frac{1}{7^3-7} - \cdots .$$

There is further 'acceleration of convergence' of this series in a graded manner* in the form

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots \pm \frac{1}{n} \pm f_i\ (n+1),\ i = 1,\ 2,\ 3,$$

where $f_1(n) = \frac{1}{2} / n$, $f_2(n) = \frac{1}{2} n/(n^2+1)$, $f_3(n) = \dfrac{[(\frac{1}{2} n)^2 + 1]}{[\frac{1}{2} n(n^2 + 4 + 1)]}$.

One traditional feature about Indian work is that only results are stated in the form of verses or sūtras and justification or proof of them is seldom given. This has led modern scholars to conclude, without consideration, that the work was borrowed from other sources. Western scholars who do not have this prejudice were/are fortunately there. For example, D. T. Whiteside of Cambridge, the editor of the sixth volume of mathematical papers of Sir Isaac Newton (in Vol. II, p. 237, footnote 122) concedes the priority of the Kerala work to Newton's. J. Hofmann of Germany and A.P. Yushkewitch of the USSR are two other scholars in the history of mathematics who have acknowledged the credit of this work. One exception to the Sūtra type of exposition is Yuktibhāṣa of Jyéstadeva (*C*.1500-1610 A.D.) of Parakroda (Parannottu) which gives proofs of the assertions in Tantrasangraha with traditional presentation but acceptable to modern students of mathematics. We owe much to the late C. T. Rajagopal** and his coworkers, who exposed the materiạl to the English-knowing world through a series of articles after a century after Whish's discovery, for having made this work of the Kerala mathematicians of yore acceptable to these western scholars. Interested readers may study these articles to find for themselves the quality and naturalness of the work mentioned above. The

* $1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \ldots + \frac{1}{49} - f_3\ (50)$ is $\frac{1}{4}\pi$ up to 11 decimal places.

** *See e.g. Scr. Math.*, 1949, **15** 201–209. Dr. K. V. Sarma has, more recently, done much bibliographical research on this theme.

work of Āryabhata and Bhāskara II was at best elementary algebra. The Kerala work relates to more sophisticated rudiments of analysis. Much needs to be done in unearthing more mathematical discoveries in these scriptures and fixing up their relevance to astronomical results. The real handicap now is the non-availability of scholars capable of comprehending traditional work in Sanskrit and presenting them in modern language. Other works relating to this theme are: Kriyākramakari of Sankara Variyar (*C.* 1500-1560 A.D.) (a commentary on Līlāvati), Yuktīdīpika (a commentary of Tantrasangraha by the same author), Karaṇapaddhati of Putumana Somayāji (*C.* 1660–1740 A.D.), Sadratnamāla of Sankara Varman (1800–1838 A. D.), the last indicating the existence of the tradition even in the 19th century.

Western education and its influence in making Indian youth disbelieve any information which is 'unscientific' (again an undefined term) and the neglect of traditional learning because of this influence has obviously killed this tradition. There can be no doubt that the Indian work right from the Vedic period was original with a vengeance. Present day mathematical work barring a few exceptions, Srinivasa Ramanujan being one, is all borrowed technology. There is a striking connection between the Kerala work and Ramanujan's work. A continued fraction expansion for

$$[\Gamma(\frac{x+1}{4})\,/\,\Gamma(\frac{x+3}{4})]^2 = \frac{4}{x+}\;\frac{1^2}{2x+}\;\frac{3^2}{2x+}\;\frac{5^2}{2x+}\;\frac{7^2}{2x+}$$

is attributed to Brouncker. Ramanujan's procedure for obtaining this expansion is a method of successive reduction used in the Kerala work[†]. Possibly Ramanujan's intuition worked in the same way his ancestors did.

Let me end with an observation relevant to improving standards in our educational system. The Indian tradition clearly points to identifying the students with aptitude and capability and imparting them knowledge available at the time and expecting them to improve on it by themselves or by discussion with fellow students based on the need of the community (for example relating to astronomy). We have lost this tradition and are putting before our youngsters ideas found in western work for being improved on. This process has not got for our nation any recognition except for individual cases of excellence. To gain international recognition for mathematical activity, it is high time that we make the talented young children of our country interact among themselves and enable them to produce 'original' work which is need-based and significant.

[†] *See* C. T. Rajagopal and M. S. Rangachari, On medieval Kerala mathematics, *Arch. History Exact Sci.*, 1986, **35** 91–99.

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Srinivasa Ramanujan: A biographical sketch and a glimpse of his work

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There have been three truly great algorists in the entire history of mathematics. An algorist is a mathematician who is known for his manipulative ability in a jungle-like algebra of symbols. For him the solution of problems of unusual kinds comes very naturally. He can devise ingenious tricks like the replacement of one or more of the variables in an equation by functions of other variables and thus while apparently complicating the problem actually can throw light on difficult situations which have defied superb intellects of the past. He can manipulate formulae involving infinite processes without being constrained by the puristic need to pay attention to rigour, convergence and mathematical existence and through his extraordinary intuition can arrive most often at the right formulae. Even when he errs the resulting formula is so elegant that his successor mathematicians spend all their time in perfecting or salvaging that beautiful false result of his. He has an unexplainable faith in his own intuition. Ramanujan was one such mathematician. The other two algorists were Leonard Euler (1707–1783) and Carl Gustav Jacobi (1804–1851). But these two had the advantage of a complete university education behind them and unlike Ramanujan they went through the trodden path of academics to acquire their reputation. But Ramanujan did not have this good fortune of a formal university training. He became a mathematician before anybody could think of training him. It is well that he was not trained early because it is debatable whether he could have been so prolific if he had been trained to watch every mathematical step of his. One may say that he was 'uneducated' if I may be permitted to use that word, compared to an Euler or a Jacobi or for that matter any mathematician in the world. Ramanujan was a self-taught genius. He could dispense with all the technical elaborations of the 18th and 19th century mathematics and still have much to say, which continues to occupy the attention of several mathematicians in the world. His dramatic rise to world recognition and his very short career of formalised activity in the then-best of the universities of the world constitute a thrilling success story, so far as the world of mathematics and the pride of India are concerned. However, but for a succession of a few accidents which can be named the world might have missed him.

Born in 1887, Ramanujan was brought up in an orthodox traditional South Indian environment. He was an enigma to his teachers even at school because of his prodigious memory and unusual mathematical talent which began to show even before he was ten. That was the age when he topped the whole district at the Primary examination and this procured him a half-fee concession at school, namely Town High School, Kumbakonam.

At the age of 12, he borrowed Loney's *Trigonometry*, Part II, from a student of the B.A. class who was his neighbour. That student was amazed to find that this young boy, about 7 to 8 years his junior, had not only finished mastering the book at one reading but he had taught himself to do every problem in it. This book, though called Trigonometry, has some of the advanced topics of mathematics in it. The treatment of these subjects is weak but the results belong to that part of mathematics called Analysis, which deals with continuous processes and expressions which grow in numerical value boundlessly. Topics such as the exponential function, logarithm of a complex variable, hyperbolic functions, infinite products and infinite series expansions of trigonometric functions are dealt with in the book. This book was Ramanujan's first contact with these advanced topics. It is an irony of fate that a better and modern treatment of these areas was not available to Ramanujan. Whittaker's *Modern Analysis* had just arrived but had not reached up to Ramanujan's environment. Bromwich's *Infinite series,* Carslaw's *Fourier series and integrals,* Pierpoint's *Theory of functions of a real variable* and Gibson's *Calculus* were just being written. If Ramanujan had had any one of these books at that age when he was gulping Loney's Trigonometry and Carr's Synopsis, would it not have made a difference in the mathematical style of Ramanujan? Perhaps. Mathematicians are divided on the answer to this question. There have not been enough geniuses in the world, on whom you can perform controlled experiments in order to answer such questions decisively!

I just mentioned Carr's Synopsis. It was this book, called *A synopsis of elementary results in pure mathematics* that created an imperishable record for itself in history, by passing through the hands of Ramanujan in his early teens. Ramanujan was captivated by its contents. It brought forth all his powers, not because it was a great book, but because it was just a compilation of about 6000 theorems with very sketchy proofs, if at all. The challenge to Ramanujan was irresistible and he started working out the proofs of results there in his own way out of his own thinking. Not only could he supply proofs to innumerable results there but he proceeded further to improve them and create his own theorems and results. He began writing theorem after theorem on the pages of quarto notebooks which are today collectively called Ramanujan's *Notebooks.*

He passed the Matriculation examination of the University of Madras in December 1903, secured a first class, and earned for himself the Subramaniam scholarship in the FA (First Examination in Arts) class at Government college, Kumbakonam. His subjects were English, Mathematics, Physiology, Roman and Greek History and Sanskrit. But Mathematics absorbed all his time and energy and he duly failed in the annual examination because of poor marks in the subjects other than mathematics and thus lost his scholarship. He left Kumbakonam, got himself lost somewhere in Andhra region, came back to Government College, Kumbakonam, after a year but could not get the necessary attendance certificate in December 1905 for the examination and thus was lost to Kumbakonam College for ever. Later, he completed the second year FA at Pachaiyappa's college, Madras and sat for the examination in December 1907. Again he failed for the same reason as before.

Prof. S. R. Ranganathan, the first Librarian of the University of Madras and a mathematician himself, calls the period 1907–11 the first period of super-activity in the life of Ramanujan, and writes: 'Inner light began to lead him. The urge for the pursuit of mathematics became irrepressible. The depression due to failure in the FA examination could not repress it. Failure to get employed could not shake it. Poverty and penury could not obstruct it. His research marched on undeterred by environmental factors– physical, personal, economic or social; magic squares, continued fractions, hypergeometric series, properties of numbers-prime as well as composite, partition of numbers, elliptic integrals and several other such regions of mathematics engaged his thought'. He had to do all this by discovering them *de novo*, because his immediate neighbourhood contained no person or book knowledgeable in these areas. He recorded his results in his notebooks. Proofs were often absent. The profoundity of contents of these notebooks as they are being analysed today reveal more and more staggering complexities. Intuition played a large part in these researches. There are three such notebooks in all, containing 212, 352 and 33 pages respectively. Exact facsimiles of these notebooks have now, since 1957, been published in two volumes by the co-operative effort of the University of Madras, the Tata Institute of Fundamental Research and Sir Dorabji Tata Trust.

It was during this period at the age of 22 that Ramanujan was married to Srimathi Janaki, then 9 years old. In 1910 Ramanujan heard of the Indian Mathematical Society which had been founded just three years earlier by Prof. V. Ramaswamy Iyer, a Deputy Collector by profession. Ramanujan ran to him at Tirukkovilur for help. To Ramaswamy Iyer goes the credit of being the first among the chain of discoverers of the genius that was Ramanujan. With his introduction Ramanujan went to Prof. Seshu Iyer and the latter put him on to Dewan Bahadur R. Ramachandra Rao, Collector of Nellore District. This historic meeting that took place in December 1910 between the genius and his patron, should be described in Ramachandra Rao's own words: 'Suspending judgment I asked him to come over again and he did. And then he had gauged my ignorance and showed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then step by step he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me'. Ramachandra Rao undertook to pay Ramanujan's expenses for a time. After a few months, being unwilling to be supported by any one for any length of time, Ramanujan accepted a clerk's appointment in the office of the Madras Port Trust. But mathematical work did not slacken. His earliest contribution to the *Journal of the Indian Mathematical Society* appeared in 1911. By this time, the Chairman of the Madras Port Trust, Sir Francis Spring, also took interest in him. The clerk in the Madras Port Trust office had become the subject of talk in the academic circles of Madras. Several attempts were made to get for him a regular scholarship from the University of Madras. Mr. R. Ramachandra Rao, Prof. C. S. T. Griffith of the Madras Engineering College, Prof. M. G. M. Hill, University College, London to whom some of Ramanujan's results were communicated, Dr. Gilbert Walker, a senior Wrangler and then Head of the India Meteorological Department, Prof. B. Hanumantha Rao, Chairman of the Board of Studies of the University of Madras, and Justice P. R. Sundaram Iyer—all had a role to play in the succession of events that finally brought Ramanujan to the University of Madras as a Research Scholar on May 1, 1913 at the age of 26 on a stipend of Rs. 75 per month.

Ramanujan thus became a professional mathematician and remained as such for the rest of his short life. He was now above want and had the academic setting to work on his mathematics. Upon the suggestion of Prof. Seshu Iyer and others Ramanujan began a correspondence with Prof. G. H. Hardy, then Fellow of Trinity College, Cambridge. His first historic letter to Prof. Hardy in January 1913, contained an attachment of 120 theorems, all originally discovered by him. Prof. Hardy's first reaction was to dismiss the letter. But later in the evening he and Prof. Littlewood spent two to three hours on the results in the letter. Several of the results completely floored the two experts. Even assuming that some of them were wrong they could not think of any other explanation but that here was a serious mathematical mind, though uninformed. They decided that the author was not a crack, but a genius. History was made in that decision. They decided to encourage Ramanujan. Their efforts to bring him to England finally materialised in March 1914.

Ramanujan spent four very fruitful years at Cambridge, fruitful certainly to him, but more so to the world of mathematics. Hardy records that the time he spent with Ramanujan from 1914 to 1918 was one of the 'most decisive events' of his life-Hardy's life, that is. Later when Ramanujan died at the unexpected age of 32, Hardy in trying to assess Ramanujan's mathematical work before he arrived in England, found it difficult to conclude whether Ramanujan had been aware of the mathematics contained in such and such well-known books. Hardy regrets that he could have easily asked Ramanujan these biographical questions in a straight-forward manner and 'Ramanujan would have answered them frankly'. But says Hardy, that was not to be. Hardy thought it ridiculous at the time to keep asking whether he had seen this book or that while 'he was showing me half a dozen or more new theorems each day': such was the prolific nature of Ramanujan's creativity. Prof. Hardy did try to 'teach' Ramanujan some of the existing mathematics which 'he ought to know', but Hardy was always in doubt whether by 'teaching' Ramanujan he was doing the right thing or not, to the genius in him. This period of Ramanujan has been well chronicled and suffice it to say that out of this superactivity, Ramanujan published 27 papers, seven of them jointly with Hardy. In 1918, he was elected Fellow of the Royal Society and in the same year was also elected Fellow of Trinity College, both honours coming as the first to any Indian. The University of Madras rose to the occasion and made a permanent provision for Ramanujan by granting him an unconditional allowance of £ 250 a year for five years from April 1, 1919, the date of expiry of the overseas scholarship that he was then drawing. The University was also to be moved, by Prof. Littlehailes, the new Director of Public Instruction, who had just returned from the Bombay Conference of the Indian Mathematical Society for the creation of a University Professorship of Mathematics and Ramanujan to be offered that Professorship, but alas, fate decided otherwise.

Unfortunately, Ramanujan had to spend the fifth year of his stay in England in nursing homes and sanatoria. He returned to India in April 1919 and continued to suffer his incurable illness. All the time his mind was totally absorbed in his mathematics. Thus arose the so-called *Lost Notebook* of Ramanujan, which has been discovered in the last decade. It contains 100 pages of writing and has in it a treasure-house of about 600 fascinating results. Prof. G. E. Andrews of Pennsylvania State University has started writing a series of papers editing this Lost Notebook of Ramanujan. Ramanujan's discoveries and flights of intuition contained in the four notebooks and in his 32 published papers as well as in the three Quarterly Reports which he submitted to the University of Madras in 1913–14, have thrilled mathematicians the world over. More than two hundred research papers have been published in the world as a result of his discoveries. We shall therefore end this account of Ramanujan by attempting to record some glimpses of his mathematical achievements, even though in a simple and sketchy manner.

A partition of an integer N is a finite sequence $a, b, c, d, \ldots, r$ of positive integers, called 'parts' of the partition, such that

$$a + b + c + \ldots + r = N.$$

For example, 4, 3, 3, 2, is a partition of 12. We write the partition as 4332 without even the commas separating the integers. 522111 is another partition of 12. Note that we always write a partition in such a way that as we read it, the parts do not increase. How many partitions are there of a given integer n? The answer is $p(n)$ in standard terminology.

$p(1) = 1$

$p(2) = 2$, for 2 and 11 are the partitions of 2.

$p(3) = 3$, for 3, 21 and 111 are the partitions of 3.

$p(4) = 5$, for 4, 31, 22, 211 and 1111 are the partitions of 4.

And so on. $p(200) = 397299029388$. Thus $p(n)$ becomes very large very rapidly.

Very little is known about the arithmetical properties of $p(n)$. Even questions like whether $p(n)$ is odd or even, for a given n, is difficult to answer. Ramanujan was the earliest mathematician to enquire into such properties. Ramanujan observed properties like the following: Whatever integer n might be, $p(5n + 4)$ is divisible by 5; $p(7n + 5)$ is divisible by 7 and similar ones. In connection with these properties, Ramanujan proved a number of identities, one of which is:

$$p(4) + p(9)\,x + p(14)\,x^2 + \ldots = \frac{5\,\{(1-x^5)\,(1-x^{10})\,(1-x^{15})\,\ldots\}^5}{\{(1-x)\,(1-x^2)\,(1-x^3)\,\ldots\}^6}.$$

This result has been considered to be representative of the best of Ramanujan's work by Hardy. Hardy says: 'If I had to select one formula for all Ramanujan's work, I would agree with Major Macmahon in selecting the above'.

The practical evaluation of $p(n)$ was, till the time of Ramanujan, done by a formula which goes back to the Euler-Jacobi tradition but was applicable to only small values of n. In 1918 Hardy and Ramanujan published a joint paper in the *Proceedings of the London Mathematical Society* on an exact formula for $p(n)$. It is very complicated to write for a lay audience. It is considered as a crowning achievement in the theory of partitions. The astonishing part of the discovery and establishment of this theorem is that, we can say, on the authority of the collaborator, Prof. Hardy, we would be nowhere near the final theorem as it is known today but for Ramanujan's unusual intuition and stroke of insight, at two stages of breakthrough in the evolution of the theorem.

In the 120 theorems that he sent to Hardy, in his first letter from India, Ramanujan claimed as one of his theorems that the number of primes less than x is

$$\int_c^x \frac{dt}{\log t} - \frac{1}{2}\int_c^{\sqrt{x}} \frac{dt}{\log t} - \frac{1}{3}\int_c^{\sqrt[3]{x}} \frac{dt}{\log t} - \frac{1}{5}\int_c^{\sqrt[5]{x}} \frac{dt}{\log t}$$

$$+ \frac{1}{6}\int_c^{\sqrt[6]{x}} \frac{dt}{\log t} - \ldots, \text{ where } c = 1.45136380 \text{ nearly.}$$

Ramanujan, of course, had not merely guessed his theorems such as this. No flight of imagination could rise to such heights and to such precision. Actually the above result of Ramanujan is false, as shown elaborately by Hardy in his 'Lectures on subjects suggested by the life and work of Ramanujan'. However, the very fact that Ramanujan discovered the above series, beter known to mathematicians as Riemann's series, all by himself is a stroke of genius and 'a very astonishing performance'. The error in Ramanujan's statement involves subtleties of complex function theory, a nineteenth century development in the mainstream of mathematics, of which he was not aware until Hardy ventured to 'teach' him. The question about prime numbers that Ramanujan seeks to answer in the above theorem is one of the most fascinating in all of mathematics and the very fact that Ramanujan, as a mere untutored explorer, could rise to the heights of the maturity of Riemann, is Intuition Par Excellence!

Next we shall refer to Entry 29 of Chapter V of Ramanujan's second *Notebook*. It is actually an entry cancelled by Ramanujan himself; and there lies the interest in this story. The entry reads as follows:

$$\frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})\ldots}$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^{2+3}}{(1-x)(1-x^2)} + \frac{x^{2+3+5}}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \frac{x^{2+3+5+7}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \ldots$$

Let us explain the genius of Ramanujan in cancelling this entry. Suppose both the above expressions are expanded in powers of x, we may then have

$$1 + c_2x^2 + c_3x^3 + c_4x^4 + \ldots = 1 + d_2x^2 + d_3x^3 + d_4x^4 + \ldots.$$

One can calculate c_2 and d_2 and see that they are equal. Similarly $c_3 = d_3$; $c_4 = d_4$ and so on. A natural temptation is to generalise and question whether $c_n = d_n$ for every integer

n. At this point it is necessary to digress and warn the non-mathematical reader on a culture that is unique to a mathematical mind and discipline. Consider the following statement:

(*): n is not divisible by both 2^8 and 5^8.

This statement (*) is true for all numbers $n = 1, 2, 3, 4, \ldots$ up to $n = 99{,}999{,}999$. For experimental scientists a rule which is valid for such a large number of cases, is valid for 'all practical purposes' as a general rule. But, for a mathematician, (*) is not true for all n; for it fails for $n = 10^8$ and all multiples of 10^8.

Thus, when Ramanujan wrote the above formula he immediately struck it off as not true, because surprisingly, though

$$c_n = d_n \text{ for } n = 1, 2, 3, \ldots, 20$$

it happens that

$$c_{21} = 30;\ d_{21} = 31 \text{ and so } c_{21} \neq d_{21}, \ldots.$$

So the two sides of the formula are not equal! But then why did he write it all, in the first place? My only guess is that he did not work out his formula on a separate sheet of paper and then transfer it to his notebook for then he would not have started writing the formula at all. As he wrote it thinking it to be true, he must have had his own methods of verifying the truth mentally and by the time he finished writing it, he must have realised the falsity of the formula and he must have struck it off!

Incidentally even this entry has generated much research. Two sequences $\{c_n\}$ and $\{d_n\}$ are said to be a 'Ramanujan pair' nowadays if $\{c_n\}$ takes the place of 2, 3, 5, 7, 11, ... on the left hand side of the above formula and $\{d_n\}$ takes the place of 2, 3, 5, 11, ... on the right hand side and the two sides are equal. As of today it has been proved that there are on the whole only 10 Ramanujan pairs out of a theoretical possibility of an infinite number of pairs. This rarity of Ramanujan pairs shows how Entry 29 almost tantalised Ramanujan to write it and then cancel it in no time.

Our last example is an amusingly delightful entry from the *Lost Notebook* of Ramanujan, not very difficult to visualise for the scientific layman:

$$\frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{b^n}{(1-ax^n)\,(1-ax^{n-1}y)\,(1-ax^{n-2}y^2)\,\ldots\,(1-ay^n)}$$

$$= \frac{1}{1-b} \sum_{n=1}^{\infty} \frac{a^n}{(1-bx^n)\,(1-bx^{n-1}y)\,(1-bx^{n-2}y^2)\,\ldots\,(1-by^n)}.$$

Prof. G. E. Andrews in his comments on this entry, gives an analytic proof of this using very heavy mathematics initiated by Ramanujan himself. In the same breath Andrews also observes that there is a combinatorial proof of the above using ideas of symmetry. Which method did Ramanujan use to arrive at this formula, or did he use a third method? These are questions for which we may never know the answer.

Such was Ramanujan and such was his genius. One may ask: In what sense is his mathematics relevant? Shall one reply that R. J. Baxter of the Australian National University has found that some of Ramanujan's work was exactly what he needed to solve the hard hexagon model in statistical mechanics? Or shall one quote Carlos Moreno of the City University of New York that Ramanujan's work in the area of modular forms is exactly what physicists need when they work on the 26-dimensional mathematical models of string theory? No. The question about relevance is irrelevant, as far as an assessment of Ramanujan's work is concerned. Ramanujan is great not because his work can also be used in modern technology but because his ideas and innovative genius have not been surpassed ever before or even 100 years after him. William Gosper of Symbolics, Inc., while recently devising a new computer algorithm to calculate π for 17.5 million digits finds that his best ideas had already been discovered by Ramanujan. We shall only quote B. C. Berndt who has just completed editing and analysing the 21 chapters of Ramanujan's second *Notebook* — Part I of Berndt's analysis was published by Springer-Verlag in 1985 — and this will give the layman a quick glimpse of the phenomenon that was Ramanujan:

> 'Because of the unique circumstances shaping Ramanujan's career, inevitable queries arise about his greatness. Here are three brief assessments of Ramanujan and his work.
>
> 'Paul Erdös has passed on to us Hardy's personal ratings of mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100. Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100.
>
> 'Neville began a broadcast in Hindustani in 1941 with the declaration, "Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years.
>
> 'In notes left by Wilson, he tells us George Polya was captivated by Ramanujan's formulae. One day in 1925 while Polya was visiting Oxford, he borrowed from Hardy his copy of Ramanujan's *Notebooks*. A couple of days later, Polya returned them in almost a state of panic explaining that however long he kept them, he would have to keep attempting to verify the formulae therein and never again would have time to establish an original result of his own.
>
> 'To be sure, India has produced other great mathematicians, and Hardy's views may be moderately biased. But even though the pronouncements of Neville and Hardy and overstated, the excess is insignificant, for Ramanujan reached a pinnacle scaled by few'.

Ramanujan's birth, his super activity in Madras and Cambridge, his glorious rise and his unfortunate death — all seem to have happened in a flash. He came and went like a meteor. When comes such another?

Madras [223]

5th Aug. 1913.

From S. Ramanujan, Scholarshipholder in Mathematics.

To The Board of Studies in Mathematics.

Through The Registrar, University of Madras.

Gentlemen,

With reference to para. 2 of the University Registrar's letter no. 1631 dated the 9th April, 1913, I beg to submit herewith my quarterly Progress Report for the quarter ended the 31st July, 1913.

The Progress Report is merely the exposition of a new theorem I have discovered in Integral Calculus. At present there are many definite integrals the values of which we know to be finite but still not possible of evaluation by the present known methods. This theorem will be an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated. For instance, the integral treated in Ex. (v) note Art. 5 in the paper, Mr. G. H. Hardy, M.A., F.R.S., of Trinity College, Cambridge, considers to be "new and interesting". Similarly the integral connected with the Besselians

[224]

Function of the n^{th} order which at present requires many complicated manipulations to evaluate can be readily inferred from the theorem given in the paper. I have also utilised this theorem in definite integrals for the expansion of functions which can now be ordinarily done by Lagrange's, Bürmann's, or Abel's theorems. For instance, the expansions marked as examples nos. (3) and (4) Art. 6, in the second part of the paper.

The investigations I have made on the basis of this theorem are not all contained in the attached paper. There is ample scope for new and interesting results out of this theorem. This paper may be considered the first instalment of the results I have got out of the theorem. Other new results based on the theorem I shall communicate in my later reports.

I beg to submit this, my maiden attempt, and I humbly request that the Members of the Board will make allowance for any defect which they may notice to my want of usual training which is now undergone by College Students and view sympathetically my humble effort in the attached paper.

I beg to remain,

Gentlemen

Your obedient servant,

S. Ramanujan.

1. Subject of the paper. [225]

If $F(x)$ be a function capable of expansion in positive integral powers of x, then

(A). The value of $\int_0^\infty x^{n-1} F(x)\,dx$ can be found from the coefficient of x^n in the expansion of $F(x)$, and conversely.

(B) The expansion of $F(x)$ in powers of x can be found if the value of the integral $\int_0^\infty x^{n-1} F(x)\,dx$ be known.

2. A:–

Let the expansion of $F(x)$ be

$$\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots ,$$

Then $$\int_0^\infty x^{n-1} F(x)\,dx = \Gamma(n)\,\phi(-n).$$

Dem. We know $$\int_0^\infty e^{-mx} x^{n-1}\,dx = \frac{\Gamma(n)}{m^n}.$$

By giving the values $1, r, r^2, r^3, \cdots$ to m on both the sides, multiplying the results by

$$f(a),\ \frac{h f'(a)}{1!},\ \frac{h^2 f''(a)}{2!},\ \frac{h^3 f'''(a)}{3!}, \cdots ,$$

and adding up all these results, we have

$$f(a)\int_0^\infty e^{-x} x^{n-1}\,dx + \frac{h f'(a)}{1!}\int_0^\infty e^{-rx} x^{n-1}\,dx + \frac{h^2 f''(a)}{2!}\int_0^\infty e^{-r^2 x} x^{n-1}\,dx + \frac{h^3 f'''(a)}{3!}\int_0^\infty e^{-r^3 x} x^{n-1}\,dx + \cdots$$

$$= \Gamma(n)\left\{ f(a) + \frac{h}{r^n}\,\frac{f'(a)}{1!} + \frac{h^2}{r^{2n}}\,\frac{f''(a)}{2!} + \frac{h^3}{r^{3n}}\,\frac{f'''(a)}{3!} + \cdots \right\}.$$

[226]

Expanding $e^{-x}, e^{-rx}, e^{-r^2x}, \dots$ on the left side in ascending powers of x and collecting all the terms that contain the same power of x, we have, by applying Taylor's theorem,

$$\int_0^\infty x^{n-1}\left\{f(a+h) - \frac{x}{1!}f(a+rh) + \frac{x^2}{2!}f(a+r^2h) - \cdots\right\}dx = \Gamma(n)\, f\!\left(a+\frac{h}{r^n}\right).$$

Now let us suppose $f(a+hr^n) = \phi(n)$, treating a, h and r as constants. Then we see that $f\left(a+\frac{h}{r^n}\right) = \phi(-n)$; and also $f(a+h), f(a+rh), f(a+r^2h), \dots$ are respectively equal to $\phi(0), \phi(1), \phi(2), \dots$ Substituting these results in the above we have

$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \cdots\right\}dx = \Gamma(n)\,\phi(-n). \qquad \text{Q.E.D.}$$

3. <u>When</u> <u>valid?</u>

The above theorem is legitimate if the following conditions are satisfied.

(a). As already stated, $F(x)$ should be capable of expansion in positive integral powers of x.

(b). $F(x)$ should be finite and continuous between the limits 0 and ∞, but not necessarily at 0 and ∞.

[227]

(c). n should be positive

(d). $x^n F(x)$ should vanish when x becomes infinite.

The first two conditions are evident from the nature of the integral itself.

The third condition is necessary because we have used the Eulerian integral $\int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)$, which is true only when n is positive.

The fourth condition is also necessary; for if, when $x=\infty$, $x^n F(x)$ does not vanish but be finite, say equal to a, then the greatest term in the expansion of $x^{n-1} F(x)$ is a/x, and consequently the greatest term in $\int x^{n-1} F(x) dx$ is $a \log x$, which is infinite when $x = \infty$. Hence we see that if, when $x=\infty$, $x^n F(x)$ is finite, then

$$\int_0^\infty x^{n-1} F(x) dx$$

is infinite; and much more so it will be if $x^n F(x)$ is itself infinite when x becomes infinite.

Although the first three conditions are necessary in case of oscillating functions, such as the circular, Besselian and other functions, yet the fourth condition differs for different functions we take.

[228]

4. Generalization :-

(a). The theorem can be used not only in case of Integrals having the limits 0 and ∞, but also in case of Integrals having any two limits; for any integral $\int_\alpha^\beta \psi(x)dx$ may be transformed to an Integral of the form $\int_0^\infty F(x)dx$ by suitable substitutions such as $\frac{x-\alpha}{\beta-x}=y$, etc.

(b). According to the condition 3(a), $F(x)$ may include all algebraic functions and all transcendental functions which can be expanded in ascending powers of x, such as $\cos x$, $\sin x$, e^{-x}, $\tan^{-1}x$, $\log(1+x)$, etc. but if $F(x)$ contains transcendentals of the form $\log x$, etc., which cannot be expressed in powers of x, we can substitute e^y, etc. for x and then apply our theorem.

(c). Similarly, by suitable substitutions, all fractional powers also may be removed.

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Ramanujan's *Notebooks*

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1. G. N. Watson who had done so much to elucidate many aspects of Ramanujan's work and who did such pioneering work on Ramanujan's conjectures on the partition function $p(n)$ and the still not very well understood mock-theta functions of Ramanujan, was the first person to give a glimpse of the many beautiful formulae contained in Ramanujan's *Notebooks*. As Watson[1] said "Nobody who has made the acquaintance of any of Ramanujan's work, will find the least difficulty in believing that the scope and variety of the material contained in the *Notebooks* are sufficiently extensive". It would be impossible therefore to give anything approaching a complete account of them even in a moderately lengthy paper; far less in a short essay like the present one.

In the early thirties Watson and B. M. Wilson decided to edit the *Notebooks* of Srinivasa Ramanujan and publish them. Wilson was assigned the task of writing on the earlier chapters and Watson that of the later chapters of the *Notebooks*. Unfortunately B. M. Wilson died on March 18, 1935, at the age of 38, of a blood infection following a hernia operation. Watson, after writing more than two dozen papers on Ramanujan's work and after making significant advances in editing the *Notebooks*, seems to have lost interest, around 1940, in Ramanujan's work and so the Notes left by Wilson and Watson, though extensive were, in many respects, incomplete. About ten years back, Bruce Berndt of the University of Illinois, Urbana, USA undertook the challenging task of completing the work of B. M. Wilson and Watson and getting the edited versions of the *Notebooks* published. One volume of the these Notes is already out and two more volumes are to appear soon. Those, who, as Watson says, desire to make acquaintance with the *Notebooks* of Ramanujan, will find in the Bruce Berndt volumes, proofs of all the statements Ramanujan made in the *Notebooks*. A study of these, it is to be hoped, will inspire young Indian mathematicians to find new insights into and connections of Ramanujan's ideas in modern mathematics and perhaps physics too.

2. Ramanujan, as is wellknown by now, was born in Erode in Tamil Nadu on December 22, 1887 to poor parents Srinivasa Iyengar and his wife Komalathammal. Srinivasa Iyengar was a modest clerk in a cloth merchant's shop at Kumbakonam earning a poor salary of rupees twenty a month. Ramanujan's father had two more sons born to him after Ramanujan and the large family of seven including the grandparents of Ramanujan found it very hard to manage the house at Kumbakonam. Still Ramanujan

joined the Town High school in Kumbakonam where his friends and teachers alike wondered at his prodigious memory. It is said that he could give the values of $\sqrt{2}$, e, $\pi, \ldots$ to a large number of decimal places. In the high school he learnt for himself trigonometry and was far ahead of his class in mathematics. In the sixth form or so, a friend of his borrowed for him Carr's *Synopsis of pure mathematics* from the local college library. Carr's book was a modest one containing statements of more than four thousand theorems spread over a large area of college mathematics, with proofs which are little more than cross references. Strangely enough this was the book that was to trigger his genius. From the time he got this book, his whole world seemed to revolve around verifying the statements in the book and in many cases generalizing them too. He also seems to have recorded these statements in a notebook, which unfortunately are not available now. In 1903, Ramanujan passed the matriculation examination of the Madras University creditably enough to be awarded a scholarship for study in the Government College, Kumbakonam which was then known as the "Cambridge of South India". However all available time, at college, in all the classes irrespective of whether they were devoted to English or Tamil, Physics or Physiology, was taken up by Carr's book and mathematics. Such singleminded and intense concentration on mathematics to the exclusion of all other subjects resulted, predictably, in his failure to be promoted to the second year class; moreover he lost the scholarship that sustained him. This was in 1904 when Ramanujan was sixteen years old. From then on it became a struggle between his genius and the "inelastic and incompetent" educational system that prevailed then (and may be now too?) in South India. The genius, of course, lost the struggle. Ramanujan made three more attempts at passing the F.A. examination and failed. In 1907 he finally gave up the chase after the elusive F.A. and decided to seek a job. He was now twenty years old, extremely brilliant in mathematics and miserably poor. His mathematical work continued, however. Further he got married in the middle of 1909 and it became absolutely necessary for him to find a job and assist his father in running the large family.

3. It was at this time or sometime earlier that he began writing the *Notebooks* perhaps to show to his prospective employers, his work in mathematics, which seemed to be, in the absence of a university degree – in a highly degree-conscious society – the only letter of recommendation for him for securing a job. He started writing the first *Notebook* in which he divided the researches of the past few years into chapters, beginning with a short chapter on "Magic squares". He initially wrote on one side of a paper only. Thus the first 134 pages comprise 16 chapters. He wrote in the style of Carr, making only statements without any explanation or proof. He chose this style of presentation, perhaps, because with his phenomenal memory, he knew he could recall proofs whenever he wanted. Further, it could be that by this means he would save paper which would be useful for recording more and new theorems, which were storming his mind all the time. In this notebook which we shall call *Notebook I* there are 16 chapters upto page 134, all written on the right hand side pages only. After this Ramanujan writes on both sides of the paper and furthermore the division into chapters is given up. The next 100 pages, or so, are filled in with beautiful formulae on hypergeometric series, continued fractions, singular moduli, etc., all piled up without order just as they came to his mind. What is more, the flood of new results seems to have made him write even on pages

which he had earlier left blank. One sees a medley of first rate formulae in which results concerning hypergeometric series or q-series jostle with those on continued fractions or elliptic integrals. To study them gives a beautiful idea of the working of his mind and the evolution of his ideas on many topics in analysis. This is one of the very fine characteristics of *Notebook I*.

Finding that *Notebook I* was, perhaps, too much cluttered up with results not very well organized or that some friends desired that another copy should be made for the sake of safety, if the original is lost, Ramanujan began writing *Notebook II* which may be considered as a revised and enlarged edition of *Notebook I*. This book was started sometime in 1911 or slightly earlier (look at the last few pages and pages 30, 31 of *Notebook I* containing results in Bernoulli Numbers which formed part of the first paper published by Ramanujan in the *Journal of the Indian Mathematical Society*, 1911). Most, but not all, of the material contained in *Notebook I* is given in *Notebook II* at appropriate places. *Notebook II* has 21 neatly arranged chapters written on both sides of the paper. In this *Notebook* too while chapter XXI starts off well, after a few pages the division into chapters is given up and a host of very nice results on modular equations, singular moduli, continued fractions, etc., are stated, some results related to those in the earlier chapters and others entirely new.

There is also a short third *Notebook* most of whose pages are left blank. In the last few pages a number of beautiful results regarding singular values of modular functions related to the class number one problem are given.

In 1913, Ramanujan held a research scholarship in mathematics at the University of Madras. In 1914, mainly through the efforts of G. H. Hardy and E. H. Neville of Cambridge University, England and Col. G. T. Walker, Director-General of the Meteorological Survey of India at Simla and Sir Francis Spring, Chairman of the Madras Port Trust – where Ramanujan worked as a clerk since 1912 – Ramanujan was enabled to go to England and do research under congenial mathematical surroundings and with no financial difficulties. Ramanujan took his three *Notebooks* with him to England and when he returned in 1919 to India, he left with Hardy *Notebook I*. In 1923, three years after Ramanujan's death, Hardy published an overview of chapters XII and XIII of *Notebook I* which dealt with hypergeometric series. Hardy's account showed that Ramanujan had rediscovered many of the fundamental theorems on hypergeometric series including the theorems of Dougall, Saalschutz, Dixon, Clausen, Kummer, Thomas and others and discovered many new ones too. This short but nice account must have whetted the appetite of many people to have a look at the *Notebooks*. It was a longtime, however, before the *Notebooks* were made available to the mathematical public. Hardy sent back to the Madras University *Notebook I* which Ramanujan left with him. At the instance of Hardy and Watson, all the three *Notebooks* were transcribed and a copy was sent to Watson for his personal use. The *Notebooks* themselves were, however, lying in the vaults of the Library of the Madras University without anybody having access to them until they were resurrected and made available to the mathematical public in 1957 by the Tata Institute of Fundamental Research, Bombay who brought out a facsimile, in two volumes, of the three *Notebooks*, the second volume comprising *Notebooks II* and

III. H. J. Bhabha, Director of the Tata Institute brought the *Notebooks* from the Madras University, K. Chandrasekharan, Professor of Mathematics at that Institute, oversaw the facsimile through the Press and R. D. Choksi, Chairman of the Sir Dorabji Tata (Educational) Trust met the entire cost of the production of the facsimile. The *Notebooks* were thus published unedited; in spite of it, however, this publication was a great service to the mathematical community in the world.

As mentioned earlier Watson was the first to write about the contents of the *Notebooks*. In 1978, Bruce Berndt[2] published an essay on Ramanujan's *Notebooks* referring to the facsimile editions of the *Notebooks* made available in 1957. In 1981, R. A. Rankin[3] made some very interesting remarks on the *Notebooks* without going into the mathematics contained in them. As time passes and our knowledge of Ramanujan's work in general and the *Notebooks*, in particular, increases and matures, our perception of the beauty and importance of the results in the *Notebooks* grows and so does our ability to see how his mathematics fits in with aspects of modern mathematics. It may, perhaps, make us feel that Ramanujan's work has the "simplicity and inevitableness" which Hardy regarded as hallmarks of great work.

We shall attempt here to highlight a few results on topics which have been barely touched upon by Watson and Berndt so that, together with their accounts of the *Notebooks* one might have a larger view of the contents of the *Notebooks*. While *Notebook II* is the most important, it is often interesting and instructive to look into *Notebook I* to know how Ramanujan's ideas evolved.

4. We shall start with chapters XII and XIII of *Notebook I* which correspond to chapters X and XI of *Notebook II*. Hardy says that he had subjected those two chapters of *Notebook I* to a searching analysis. In these two chapters there are some theorems on transformations of hypergeometric series like those due to Thomas in his paper in 'Crelle' 1879. It is thus a great surprise to see that Ramanujan suddenly pops up a beautiful theorem on the left hand side page 126 of Volume I* which is a limiting case of a famous theorem due to Whipple proved in his paper of 1925 more than 14 years after *Notebook I* was written by Ramanujan. The theorem (in the notation of Bailey's Tract[4]) is:

$$
{}_6F_5\left[\begin{matrix} a, 1+(a/2), b, c, d, e \\ a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix}; -1\right] =
$$

$$
= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\ {}_3F_2\left[\begin{matrix} 1+a-b-c, d, e \\ 1+a-b, 1+a-c \end{matrix}; 1\right] \tag{1}
$$

which is a transformation of a well-poised (in the sense of Whipple) series with special form of the second parameter, into a ${}_3F_2$. Of course one can see that it is a limiting case of Whipple's famous theorem or transformation of a well-poised ${}_7F_6$ into a ${}_4F_3$. In fact

* All references to page numbers are to the facsimile edition of the *Notebooks*.

Whipple gives in his paper (see references in Bailey's Tract[4]) the proof of statement (1) too. But why did Ramanujan suddenly, after finishing chapters XII and XIII, state (1) instead of the full-blooded transformation of Whipple which was equally easy to prove by induction?

It might be of interest to note that when Ramanujan sent his second letter of February 13, 1913 to Hardy, he sent him the summation

$$1-5\cdot\left(\frac{1}{2}\right)^5+9\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^5-13\cdot\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^5+\ldots=\frac{2}{(\Gamma(3/4))^4} \tag{2}$$

which requires precisely (1) for its proof; namely put $a=b=c=d=e=\frac{1}{2}$ and use Dixon's theorems (rediscovered by Ramanujan's *Notebook I*, p. 175, Entry 9) for the sum on the right of (1). Hardy proved (2) using Legendre polynomials and Whipple, of course, used his ${}_7F_6$ theorem. This result (2) is *not* in the *Notebooks* but seems to have been manufactured just for the letter to Hardy. Ramanujan had a beautiful eye for the novel and unusual.

In the chapters that constitute the 134 right hand side pages of *Notebook I*, there are no theorems regarding q-series. However pages 126–128, the left hand side pages left empty initially, are filled with beautiful theorems some of which go back to Heine. Was Ramanujan contemplating q-generalizations of theorems on ordinary hypergeometric series which he had given in chapters XII and XIII of *Notebook I*? These generalisations and more constitute some of the contents of Chapter XVI which is one of the most beautiful chapters of *Notebook II*. In it he again gives the limiting case of a q-generalisation of Dougall's theorem due to Jackson proved in 1921. As Jackson says L. J. Rogers, before everybody else, had proved this limiting case in 1895. Ramanujan's statement

$$\begin{aligned}&{}_6\varphi_5\left[\begin{matrix}a/q,\ \sqrt{aq},\ -\sqrt{aq},\ 1/b,\ 1/c,\ 1/d\\ \sqrt{a/q},\ -\sqrt{a/q},\ ab,\ ac,\ ad\end{matrix};q;abcd\right]=\\&=\frac{(a)_\infty(abc)_\infty(abd)_\infty(acd)_\infty}{(ab)_\infty(ac)_\infty(ad)_\infty(abcd)_\infty}.\end{aligned} \tag{3}$$

is Entry 5, Page 193, chapter XVI of *Notebook II* but is already in *Notebook I*, page 130. In fact in Entry 7 of the same chapter Ramanujan states a limiting case of Watson's q-generalisation of Whipple's ${}_7F_6$ transformation. Why did Ramanujan give only limiting cases of theorems when the original results themselves are equally simple to prove? Watson's generalisation proved in 1929 was designed to give an alternative proof of the Rogers–Ramanujan identities, which are themselves given in chapter XVI of *Notebook II* and already in *Notebook I*, page 160.

Before leaving chapter XVI let us make a few remarks on the continued fraction

$$\frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\ldots,\ q=e^{-2\pi\alpha},\ \alpha>0 \tag{4}$$

which was first discovered by L. J. Rogers. Ramanujan rediscovered this (see *Notebook I*, pages 146 and 160). His evaluation of this continued fraction for various values of α and general study of this and related continued fractions is due entirely to Ramanujan and shows his mastery of the theory of elliptic functions and complex multiplication. Indeed, in his first letter of January 16, 1913 to Hardy, he states among others, the beautiful result:

$$\frac{e^{-2\pi/5}}{1+}\ \frac{e^{-2\pi}}{1+}\ \frac{e^{-4\pi}}{1+}\ \cdots = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{5}$$

which is in chapter XVI, page 204. However in his second letter, he gives another beautiful result which is *not* in the *Notebooks* but seems to have been manufactured like (2) just for the occasion. It is slightly deeper than (5) and its statement is:

$$\frac{e^{-2\pi/\sqrt{5}}}{1+}\ \frac{e^{-2\pi\sqrt{5}}}{1+}\ \frac{e^{-4\pi\sqrt{5}}}{1+}\ \cdots = \frac{\sqrt{5}}{1+\sqrt[5]{(5^{3/4}[(\sqrt{5}-1)/2]^{5/2}-1)}} - \frac{\sqrt{5}+1}{2} \tag{6}$$

The formulae (5) and (6) which Hardy says defeated him completely, seem to have convinced him that Ramanujan was a mathematician of the front rank.

Chapters XVIII to XXI deal with modular equations, modular functions, continued fractions, singular moduli, etc. These are, of course, related and the results therein are extremely interesting and tantalizing. Ramanujan gave several forms of the modular equations of orders 3, 5, 7 . . . . They are algebraic equation connecting k and l where, in the usual notation of elliptic function theory $L'/L = n(K'/K)$ where k corresponds to K and K' and l to L and L'. However Schläfli found another type of modular equations, namely those connecting the Schläfli modular functions $f(\tau)$ and $f(n\tau)$ where

$$f(\tau) = e^{-\pi i/24}\,\frac{\eta[(\tau+1)/2]}{\eta(\tau)},\ \eta(\tau) = e^{2\pi i\tau/24}\prod_{k=1}^{\infty}(1-e^{2k\pi i\tau}), \tag{7}$$

$\tau = x+iy$, $y > 0$ and n is a positive rational number. For example if $n = 3$ one has

$$\left(\frac{f(\tau)}{f(3\tau)}\right)^6 + \left(\frac{f(3\tau)}{f(\tau)}\right)^6 = (f(\tau)f(3\tau))^3 - \frac{8}{(f(\tau)f(3\tau))^3} \tag{8}$$

which is also given by Ramanujan (Vol II, page 230, Entry 5-xii). In fact, Ramanujan gives the Schläfli modular equations of degrees 3, 5, 7, 11, 13, 17, 19 in his first *Notebook* (Vol I, p. 90). If one uses, as Ramanujan does, the formula

$$f(\tau+1) = e^{-\pi i/24} f_1(\tau),\ f_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)} \tag{9}$$

then one has formulae similar to (8) for $f_1(\tau)$ too. Ramanujan mentions this on page 90 in Volume I. This leads one to the problem of singular moduli, namely of evaluating $f(\sqrt{-m})$ and $f_1(\sqrt{-2m})$, for $m > 0$ an odd integer. It is wellknown that these are algebraic numbers and some evaluations of these are in Greenwill's book which Ramanujan seems to have studied. In *Notebook I* Ramanujan gives the evaluation of $f(\sqrt{-m})$ and $f_1(\sqrt{-2m})$ for a large number of values of m. It is really remarkable that Ramanujan who did not, as Hardy said, know the arithmetic theory of Gauss on binary quadratic forms should have selected those for which $f(\sqrt{-m})$ and $f_1(\sqrt{-2m})$ can be evaluated in terms of quadratic surds only, as for example (*Notebook I*, pages 316, 317)

$$\sqrt{8}\cdot(f_1(\sqrt{-42}))^6 = (2\sqrt{2}+\sqrt{7})(3+\sqrt{7})^3$$
$$2^{13/2}(f(\sqrt{-105}))^6 = (1+\sqrt{5})^3(1+\sqrt{3})^3(\sqrt{3}+\sqrt{7})^3(\sqrt{5}+\sqrt{7}) \tag{10}$$

Ramanujan has gone much farther than Weber, etc., in evaluating $f(\sqrt{-m})$ and $f_1(\sqrt{-2m})$ for many m. Watson, in his several papers, tries to see what methods Ramanujan could have used.

Ramanujan gives a neat method for evaluating k when $f_1(\sqrt{-2m})$ is known. This is given as a Lemma on page 320 of *Notebook I*. It says:

Let $(z^{-1/4}f_1(\sqrt{-2m}))^6 = u\cdot v,\ u>1,\ v>1$. Put

$$2U = \left(u^2+\frac{1}{u^2}\right),\ 2V = \left(v^2+\frac{1}{v^2}\right),\ W = \sqrt{u^2+v^2-1},$$

$$2S = U+V+W+1. \tag{11}$$

Then

$$k = (\sqrt{S}-\sqrt{S-1})^2(\sqrt{S-U}-\sqrt{S-U-1})^2(\sqrt{S-V}-\sqrt{S-V-1})^2$$
$$\times(\sqrt{S-W}-\sqrt{S-W-1})^2.$$

This enables one to obtain $f_1(2\sqrt{-2m})$. He has evaluated these for a large number of values of m which have the property that the field $Q(\sqrt{-2m})$ has exactly one ideal class in each genus. Not knowing the arithmetical theory of binary quadratic forms, it is possible that Ramanujan evaluated $f_1(\sqrt{-2m})$ by trial and error. He had enormous intuition and patience.

Using the relations (9) Ramanujan obtains values of $f(\sqrt{-m})$, $f_1(\sqrt{-m})$, and $f(3\sqrt{-m})$ and $f_1(3\sqrt{-m})$ in a very nice way. By the Schläfli equation one sees that $(f_1(3\sqrt{-m}))^3$ is given as the positive root of a quartic equation

$$9 = \left(1+2\sqrt{2}\frac{g_{qn}^3}{g_n^9}\right)\left(1-2\sqrt{2}\frac{g_n^3}{g_{qn}^9}\right),\ g_n = 2^{-1/4}f_1(\sqrt{-n}). \tag{12}$$

In a set of beautiful identities on pages 321 of *Notebook I* and page 241 of Volume II given by Ramanujan, which need to be investigated more closely, one obtains the beautiful formula

$$g_{81n}^3 = g_{qn} \frac{g_n^3 + \sqrt{2}\, g_{qn}}{g_{qn} - \sqrt{2}\, g_n^3} \tag{13}$$

which enables one to evaluate $g_q k_n = 2^{-1/4} f_1(3^k \sqrt{-n})$ in a relatively simple way. Because of formulae (9) one can obtain similar results for $G_n = f(\sqrt{-n}) \cdot 2^{-1/4}$. Ramanujan gives (*Notebook II* p. 296)

$$(G_{81})^3 = \frac{(2(\sqrt{3+1}))^{1/3} + 1}{(2(\sqrt{3-1}))^{1/3} - 1}. \tag{14}$$

Ramanujan's development of modular equations, singular moduli, etc., deserve to be studied much more closely than hitherto. One sees the fantastic power of his intuition and computational abilities.

There are many other aspects of elliptic function theory, for example elliptic integrals, which we leave without comment, but which deserve to be studied very closely.

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Ramanujan's work in the theory of partitions

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One could think that the mathematical lives of geographically and politically far off countries like India and Hungary could be separated very much. But this is not true. In fact, one of the first foreign mathematicians I ever heard of, was *Srinivasa Ramanujan*. My professor at the University of Budapest, *Paul Turan*, often mentioned Ramanujan's name with admiration and told us stories from the short and romantic life of this Indian genius.

Later I met many great Indian mathematicians (*e.g.* my first host in the United States was professor *R. C. Bose*) and understood that the followers of Ramanujan made Indian mathematics flourishing and very strong.

One of the fields, where Ramanujan made a major contribution was the theory of partitions. The papers written by Ramanujan (in part jointly with *Hardy*) were a breakthrough in the field. The limits in time and space do not allow us to show their proofs or to survey the modern results along these lines. Hundreds of papers were written about partitions, all of them more or less connected to the works of Ramanujan. *Andrews*[1] has written a good monograph containing some of the main directions of the theory of partitions. The present paper is just a small kaleidoscope of the field with a special emphasis on the recent Hungarian results.

Let n be a positive integer and let $p(n)$ denote the number of partitions of n into positive integer parts disregarding the order of the summands. That is, $p(n)$ is the number of integer solutions of

$$1x_1 + 2x_2 + \ldots + nx_n = n. \tag{0}$$

It is easy to find the generating function

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n$$

of $p(n)$:

$$F(x) = (1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots \tag{1}$$
$$= \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots$$

However, very little is known about the arithmetical properties of $p(n)$. Ramanujan was the first mathematician to discover such properties. The first one was[2]

$$p(5m+4) \equiv 0 \pmod 5.$$

His proof uses the famous identities of Euler and Jacobi, respectively:

$$(1-x)(1-x^2)(1-x^3)\ldots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k+1)/2} \tag{2}$$

and

$$((1-x)(1-x^2)(1-x^3)\ldots)^3 = 1-3x+5x^3-7x^6+\ldots$$
$$= (1/2) \sum_{l=-\infty}^{\infty} (-1)^l (2l+1) x^{l(l+1)/2}. \tag{3}$$

By (1) $p(5m+4)$ is the coefficient of x^{5m+5} in

$$x/(1-x)(1-x^2)(1-x^3)\ldots =$$
$$x((1-x)(1-x^2)(1-x^3)\ldots)^4/((1-x)(1-x^2)(1-x^3)\ldots)^5. \tag{4}$$

Here

$$1/((1-x)(1-x^2)(1-x^3)\ldots)^5$$

contains only powers of x^5, therefore the coefficient of x^{5m+5} in (4) is the sum of the coefficients of some x^{5m_i} in

$$x((1-x)(1-x^2)(1-x^3)\ldots)^4. \tag{5}$$

In order to prove the statement, it is sufficient to prove that the coefficients of x^{5m} in (5) are divisible by 5. Equations (2) and (3) give

$$(1/2) \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (-1)^{k+1}(2l+1) x^{1+k(3k+1)/2+l(l+1)/2} \tag{6}$$

for (5). If the exponent of a term is divisible by 5 then

$$8(1+k(3k+1)/2+l(l+1)/2)-10k^2-5 = 2(k+1)^2+(2l+1)^2 \tag{7}$$

is also divisible. The square of an integer gives a remainder 0, 1 or 4 when it is divided by 5. Therefore, the remainder of $2(k+1)^2$ is 0, 2 or 3. It is easy to see that the divisibility of (7) implies that both terms are divisible 5, consequently the coefficient $2l+1$ in (6) is really a multiple of 5, proving the statement.

He found a similar proof for

$$p(7m+5) \equiv 0 \pmod 7$$

and a more complex proof for

$$p(11m+6) \equiv 0 \pmod{11}.$$

He went further proving

$$p(25m+24) \equiv 0 \pmod{25},$$
$$p(49m+47) \equiv 0 \pmod{49}$$

and

$$p(121m+116) \equiv 0 \pmod{121}.$$

He also stated a general conjecture which turned out to be almost correct. *Krechmar*[3] proved the analogous statement for 5^3 and *Watson*[4] generalized it for 5^a. *Lehmer*[5] settled the cases 11^3 and 11^4. However, as *Chowla*[6] observed, 7^3 was a counter-example. Finally *Atkin*[7] proved the following modification of Ramanujan's conjecture:

If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$ then

$$p(\lambda) \equiv 0 \pmod{5^a 7^{[(b+2)/2]} 11^c}.$$

The other important achievement is a joint work of *Hardy* and Ramanujan[8]. It determined the asymptotic behaviour of $p(n)$. They proved that

$$p(n) \approx 1/(4n\sqrt{3}) \exp(\pi\sqrt{(2/3)}\sqrt{n}).$$

In the proof they used analytic methods. They determined the order of magnitude of the coefficients of the generating function using tools like Cauchy's theorem.

There is a very easy recurrence relation for $p(n)$:

$$np(n) = \sum_{k=1} \sum_{l=1} kp(n-kl),$$
$$p(0) = 1, \; p(-1) = p(-2) = \ldots = 0. \tag{8}$$

(8) is the sum of all partitions of n. The left hand side is obvious. To obtain the right hand side, count the number of occurrences of the integer k in the partitions. There are $p(n-k)$ partitions containing at least one k, there are $p(n-2k)$ partitions containing at least two k, ... The number of partitions containing exactly 1, 2, ... occurrences of k is $p(n-k)-p(n-2k)$, $p(n-2k)-p(n-3k)$, ... So the total number of occurrences of k is $p(n-k)-p(n-2k)+2(p(n-2k)-p(n-3k))+\ldots = p(n-k)+p(n-2k)+\ldots$ This leads to the right hand side of (8).

Hardy and Ramanujan[8] state that

$$\log p(n) \approx c\sqrt{n}$$

can be deduced from (8). Let us prove one side of this statement (on the basis of *Erdös*[9]) by induction on n:

$$p(n) < \exp(\pi\sqrt{(2/3)}\sqrt{n}). \tag{9}$$

It is obviously true for $n = 1$. By (8) and the induction hypothesis we have

$$np(n) < \sum_{\substack{k=1 \\ kl \le n}} \sum_{l=1} k \exp(\pi\sqrt{(2/3)}\sqrt{(n-kl)}).$$

Use the obvious inequality

$$\sqrt{n}-a < \sqrt{n}-\frac{a}{2\sqrt{n}} \quad (n \geq a > 0):$$

$$np(n) < \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k \exp(\pi\sqrt{(2/3)}(\sqrt{n}-kl/2\sqrt{n}) =$$

$$\exp(\pi\sqrt{(2/3)}\sqrt{n}) \sum_{l=1}^{\infty} \exp(-l\pi\sqrt{(2/3)}/2\sqrt{n})(1-\exp(l\pi\sqrt{(2/3)}/2\sqrt{n})).$$

Now it is easy to see that $\exp(-x)/(1-\exp(-x))^2 < 1/x^2$ holds for any positive x, hence we have

$$np(n) < \exp(\pi\sqrt{(2/3)}\sqrt{n}) \sum_{l=1}^{\infty} 4n/l^2\pi^2(2/3) =$$

$$\exp(\pi\sqrt{(2/3)}\sqrt{n})n(6/\pi^2)(\pi^2/6) = \exp(\pi\sqrt{(2/3)}\sqrt{n})n,$$

proving (9).

Erdös[9] succeeded in refining these elementary methods to prove

$$p(n) \approx a/n \exp(\pi\sqrt{(2/3)}\sqrt{n}),$$

without determining the exact value of a.

Since the work of Hardy and Ramanujan, the theory of probability reached a high state of development. This is why the problem is frequently considered to be a probabilistic one, nowadays. Take a variant of the original problem. Suppose that only the numbers $0 \leq x_k \leq p$ are allowed in (0) as summands. Denote by $q(p,n)$ the number of solutions of (0) under this condition. Consider the totally independent random variables ξ_k $(k = 1, 2, \ldots)$ taking on the values of $0, k, 2k, \ldots, pk$ with probability $1/(p+1)$, each. It is easy to see that the number of solutions of (0) under the condition $0 \leq x_k \leq p$ is equal to

$$\text{Prob}(\xi_1 + \ldots + \xi_n = n)(p+1)^n.$$

To determine the order of magnitude of

$$\text{Prob}(\xi_1 + \ldots + \xi_n = n) = \text{Prob}((\xi_1 + \ldots + \xi_n)/n = 1)$$

for various random variables is an important task of probability theory. The results are called *local limit distribution theorems*. The expectation is

$$A_n = M((\xi_1 + \ldots + \xi_n)/n) = (n+1)p/4.$$

So the "most probable" value of $(\xi_1 + \ldots + \xi_n)/n$ is $(n+1)p/4$. The probability of the value 1 is much less, but the speed of decrease depends on the variance

$$D_n^2 = (1/72)(n+1)(2n+1)p(p+2)/n.$$

The starting idea is that

$$\int_0^{\infty} e^{2\pi i t(1x_1 + 2x_2 + \ldots + nx_n - n)}\, dt \qquad (10)$$

is 1 iff $1x_1 + 2x_2 + \ldots + nx_n = n$, otherwise it is 0. Introduce the notation

$$f_k(t) = (1/(p+1)) \sum_{l=0}^{p} e^{itlk} \tag{11}$$

for the *characteristic function* of the random variable ξ_k. It is easy to see that

$$\text{Prob}(\xi_1 + \ldots + \xi_n = n) = \int_0^\infty e^{-2\pi itn \prod_{k=1}^{n} f_k(2\pi t)} \, dt \tag{12}$$

To obtain good asymptotic formulas we need good estimates on the so-called *trigonometric series* [11] and their products. This probabilistic approach was suggested by *Castelnuovo*[10] and surveyed in the book of *Sirazhdinov et al*[11].

$q(1, n)$ was already determined asymptotically in Hardy and Ramanujan[8] by their original method:

$$q(1, n) \approx (1/(4 \cdot 3^{1/4} \cdot n^{3/4}))\, e^{(\pi/\sqrt{3})\sqrt{n}}.$$

M. Szalay[12] found the asymptotic formula

$$q(p, n) \approx$$
$$((1-(1/(p+1)))^{3/4}/(2.6^{1/4} \cdot p \cdot n^{3/4}) \exp(2\pi/\sqrt{6}\sqrt{(1-(1/(p+1)))}\sqrt{n}).$$

Now consider an even more general question. What is the number of solutions of the diophantine equation

$$1x_1 + 2x_2 + \ldots + nx_n = N$$

under the condition $0 \leq x_j \leq p$ $(1 \leq j \leq n)$? *Joó*[13] determined an asymptotic formula for the case

$$(1-s)(1/72)n(n+1)(2n+1)p(p+2) \log n > (N - n(n+1)p/4)^2$$

On the other hand, *A. Bogmér* and *L. A. Székely*[14] asymptotically determined the number of solutions of the diophantine equation system

$$1x_1 + 2x_2 + \ldots + nx_n = N_1$$

$$x_1 + x_2 + \ldots + x_n = N_2$$

$$0 \leq x_j \leq p \quad (1 \leq j \leq n)$$

under some conditions concerning the parameters.

Finally, let me mention two applications. The first one belongs to *statistical group theory*[15]. Consider permutations belonging to the symmetric group S_n. It is easy to see that two such permutations are conjugates iff the cycle structures of them are the same, that is iff the number of cycles of length 1, 2, . . . is the same in the two permutations. Conjugacy is an equivalence relation, therefore S_n splits into equivalence classes. The number of these classes is nothing else but the number of solutions of (0), that is, $p(n)$.

In the other application, *Sali*[16] proved an extremal result for partially ordered sets. In the proof he needs the (asymptotic) number of solutions of

$$x_1 + \ldots + x_n = kn/2$$

under the condition $0 \leq x_i \leq k-1$. However, this problem is solved by Sirazhdinov *et al*[11].

I hope that this short survey proves the non-negligible influence of Ramanujan's work for the mathematics of today's Hungary.

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The circle method and its implications

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1. Srinivasa Ramanujan, in his first letter to Professor G. H. Hardy written from India in the year 1913, had tabulated a series of 15 formulae, which impressed Professor G. H. Hardy so much that he remarked "they could only be written down by a mathematician of the highest class".

Our particular concern in this lecture is the formula (1.14) which reads: The coefficient of x^n in $(1-2x+2x^4-2x^9+\ldots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right).$$

It is not clear, whether Ramanujan had a rigorous proof of this very likely not.

Even though $\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right)$ is a very good approximation to the n^{th} coefficient, it is not as good an approximation as he claimed it to be. But it seems inconceivable how he could have guessed the first approximation without ever having some sort of a proof. As Hardy remarks, "Ramanujan's false statement was one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions".

Even though the results in the partition theory are very interesting, our immediate concern here is actually the method adopted for the proof. The method is so general that it could be applied to a host of problems and thus was born one of the most fruitful methods in analytic number theory, the 'Circle method'.

2. *Partition problem*: We define $p(n)$ to be the number of partitions of n. That is, the number of ways n can be written as a sum of integers. Lest we should count the permutation of the same partition a second time, let us make a convention, that we write n as a sum of integers in the decreasing order. Thus, for example,

$$\begin{aligned}5 &= 1+1+1+1+1\\ &= 2+1+1+1\\ &= 2+2+1\\ &= 3+1+1\\ &= 3+2\\ &= 4+1\\ &= 5\end{aligned}$$

Hence $p(5) = 7$.

Now we are interested in getting an asymptotic formula for $p(n)$.

The analytic way to handle $p(n)$ is to use Euler's formula $\prod_{m=1}^{\infty} (1-Z^m)^{-1} = \sum_{n=1}^{\infty} p(n)\, Z^n$, which is easily verified by expanding the infinite product. Clearly the singularities of $f(Z) = \prod_{m=1}^{\infty} (1-Z^m)^{-1}$ are the roots of unity. Since the circle of convergence passes through the nearest singularity, we observe that the circle of convergence is $|Z| = 1$, the function is regular in $|Z| < 1$ and the circle $|Z| = 1$ is the natural boundary.

Now Cauchy's theorem on residues gives,

$$p(n) = \frac{1}{2\pi i} \int_{|Z|=r} \frac{f(Z)}{Z^{n+1}}\, dZ \quad (*)$$

where r is any positive real number < 1. Thus the seemingly combinatorial problem of tabulating the number of ways of writing n as a sum of integers, has been converted to an analytic problem of evaluating (or at any rate estimating) an integral.

We now argue roughly. If we assume $r (< 1)$ is close to 1, then the value of $f(Z)$ at $Z = r\, e^{2\pi ih/k}$ is nearly equal to the value at $Z = e^{2\pi ih/k}$), which is a pole. Consequently $|f(r\, e^{2\pi ih/k})|$ is very large. But, as we shall see later, the value of $f(r\, e^{2\pi ih/k})$ is inversely proportional to k; Hence $|f(Z)|$ is very large near the points $Z = r\, e^{2\pi ih/k}$ for small values of k, say $k \leq K$. Hence we should expect the essential contribution to the integral in (*) to come from the neighbourhood of points $Z = r\, e^{2\pi ih/k}$, $(k \leq K)$. Let us call these neighbourhoods the major arc and the rest minor arc. We shall get an asymptotic formula for the major arc and estimate the minor arc contribution to the integral. If the estimate on the minor arc is very small compared to the asymptotic formula for the major arc, then we should obtain an asymptotic formula for (*) and this is what we need.

3. We observe that if $f(Z) = \prod_{n=1}^{\infty} (1-Z^n)^{-1} = \sum_{n=1}^{\infty} p(n)\, Z^n$, then

$$f\left(\exp\left(\frac{2\pi\,\mathrm{i}(h+iZ)}{k}\right)\right) = \omega_{h,k}\cdot Z^{1/2}\exp\left(\frac{\pi(1/2-Z)}{12k}\right)$$

$$\times f\left(\exp\left(\frac{2\pi\,i}{k}\left(h'+\frac{i}{Z}\right)\right)\right) \text{— (**)}$$

Here h' is defined by $hh' \equiv 1 \pmod k$)

This is the so called 'modular' property which connects the value of $f\left(\exp\left(2\pi i\left(\frac{aZ+b}{cZ+\mathrm{d}}\right)\right)\right)$ with $f(\exp(2\pi iZ))$ for any integers a,b,c,d with $ad-bc=1$; here $\omega_{h,k}$ is a 24th root of unity. We fix a large integer N. The radius r is chosen to be $\exp\left(\frac{-2\pi}{N^2}\right)$. The major arc consists of union of small arcs around $r\exp^{2\pi ih/k}$, $(k \le N)$, the interval around $\frac{h}{k}$ being $\left(\frac{h_0+k_0}{k_0+h}, \frac{h_1+h}{k_1+h}\right)$ where $\frac{h_0}{k_0}$ and $\frac{h}{k}$ are the preceding and the succeeding entries in the Farey direction of order N.

Now note that by choosing appropriate value of Z, the left hand side of (**) becomes precisely the value of the function $f(Z)$ on the major arc. Since in this choice $|Z|$is very small, in r.h.s. of (**) we have $f\left(\exp\frac{2\pi i}{k}\left(h'+\frac{i}{Z}\right)\right)$ is nearly $i\left(\exp\frac{2\pi ih'}{k}\right)$. Thus we get a fairly good idea of $f(Z)$ which enables us to compare an asymptotic formula for the same. The way we have defined the major arc, we see that it covers the whole circle and there is no minor arc (The concept of minor arc is important, for other problems like Waring's problem). This yields ultimately

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\,k^{1/2}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{(\sinh h\,\pi/k)(\frac{2}{3}(x-\frac{1}{24}))^{1/2}}{(x-\frac{1}{24})^{1/2}}.$$

4. *Waring's problem*: As was remarked earlier, the circle method has been found useful in the other additive problems in Number Theory, that is problems about the number of representation of an integer n as a sum of integers of a special form. We shall discuss Waring's problem first.

The celebrated theorem of Lagrange, proved in 1770, asserts that every integer can be written as a sum of four squares. Thus, for example, $87 = 49+36+1+1$ and $123 =$

$121+1+1+0$. Generalizing the above result, Waring again in 1770, conjectured that every integer is a sum of 9 cubes, 19 biquadrates and so on. Let us define, following G. H. Hardy, $g(k)$ to be the minimum number of summands needed to express every integer as a sum of k^{th} powers. Thus the above would read: $g(2) \leq 4$; $g(3) \leq 9$; $g(4) \leq 19$ etc; But if we write 7 as a sum of squares we need a minimum of 4 summands, 23 needs 9 cubes to express and there is no way to express 79 using less than 19 biquadrates, thereby showing $g(2) \geq 4$; $g(3) \geq 9$ and $g(4) \geq 19$. Hence Lagrange's theorem reads $g(2) = 4$ and Waring's conjecture reads $g(3) = 9$, $g(4) = 19$ etc.

If we define $h(k) = [(\frac{3}{2})^k]2^k - 1$ where, $[x]$ as usual stands for the integral part of x; then our counter-examples 7, 23 and 79 are precisely $h(2)$, $h(3)$ and $h(4)$. On this basis, one can guess $h(k)$ is the integer which would need the maximum number of k^{th} powers. Since k^{th} powers needed for $h(k)$ is $2^k + [(\frac{3}{2})^k] - 2$, we can make the*Conjecture*: $g(k) = 2^k + [(\frac{3}{2})^k] - 2$, and this is known as Waring's problem. Incidentally, it is far from obvious that $g(k)$ is finite for every k and this was first proved by Hilbert around 1900.

In 1918 (that is almost immediately after Hardy-Ramanujan published their celebrated work on partition functions), Hardy and Littlewood observed that the circle method is capable of yielding results in Waring's problem also.

Let k be any integer. Let $f(Z) = \left(\sum_{m=0}^{\infty} Z^{m^k} \right)^s = \sum_{N=1}^{\infty} r(N)\, Z^N$ where

$r(N) = r(N,s,k)$ is the number of representations of N as a sum of s k^{th} powers. If we can prove that $r(N,s,k) > 0$ for all N, for some specific value of s, then it would follow that $g(k) \leq s$. (Our aim is to choose $s = 2^k + [(\frac{3}{2})^k] - 2$ ultimately). As before,

$$r(N) = \frac{1}{2\pi i} \int_{|Z|=r} \frac{f(Z)\, \mathrm{d}Z}{Z^{N+1}}, \text{ for some } 0 < r < 1;$$

Since the technical details are more complicated in this case, we shall not go into the details; but it suffices to say that for $k \geq 3$, $f(Z) = \left(\sum_{m=0}^{\infty} Z^{m^k} \right)^s$ does not seem to have any 'modular' property and this makes the problem more difficult. Still, using the circle method, Hardy and Littlewood succeeded in proving that $g(k)$ is finite and paved the way for the ultimate solution of the problem.

Actually one does not start with the function $f(Z)$, but a different function $G(Z)$, which differs from $f(Z)$ in two aspects, one of which was suggested by Hardy and the other by the Russian mathematician I. M. Vinogradov.

Suppose we are interested in $r(N,s,k)$, the number of ways of writing N as a sum of s k^{th} powers. Define

$$F(Z) = \left(\sum_{0 \leq m \leq N^{1/k}} Z^{m^k} \right)^s = \sum_{n=1}^{\infty} r^*(n,s,k)\, Z^n.$$

In general $r(n,s,k)$ and $r^*(n,s,k)$ are different; but for values $n \leq N$, $r(n,s,k) = r^*(n,s,k)$. The advantage in $F(Z)$ is that $F(Z)$ is a polynomial and hence holomorphic. Hence, we can as well integrate on $|Z| = 1$, instead of integrating on $|Z| = r$, with $r < 1$ and r close to 1. This simplifies the proof to a considerable extent. We define

$$G(Z) = \left(\sum_{0 \leq m \leq N^{1/k}} Z^{m^k} \right)^{s-2\lambda} \left(\sum_{u \varepsilon U} \varepsilon^{2\pi i u} \right)^2$$

where U is the set of different integers which are representable as a sum of λ k^{th} powers. Here λ, an integer, is at our choice. Then

$$G(Z) = \sum_{n=1}^{\infty} r_1(n) Z^n.$$

Here $r_1(N) \leq r(N,s,k)$. But, in practice it becomes easier to handle $r_1(N)$ and prove $r_1(N) > 0$ and deduce $r(N,s,k) > 0$. Thus

$$r_1(N) = \frac{1}{2\pi i} \int_{|Z|=1} \frac{G(Z)}{Z^{N+1}} \, dZ$$

As before, we break the integral into a major arc and a minor arc. We define the major arc as the union of small intervals around the points $e^{2\pi i h/k}$, $k \leq K$ and this works in practice (whether this is the correct choice for the major arc is a different question, which we would not go into). Then one finds an asymptotic formula for the major arc contribution and estimates the minor arc contribution. If one is lucky, one would be able to prove that the minor arc contribution is very small compared to the asymptotic formula for the major arc contribution. This gives an asymptotic formula for $r_1(N)$ and shows in particular that $r_1(N) > 0$ for large N. As far as small values of N are concerned, one can check if every such N is a sum of s k^{th} powers, using a computer.

These ideas enabled later mathematicians to completely settle Waring's problem and we shall give below some important names. The value of $g(k)$ was calculated by S. S. Pillai and L. E. Dickson independently, and almost simultaneously, for $k \geq 7$. Their formula coincides with $2^k + [(\frac{3}{2})^k] - 2$ if a certain conjecture (which is almost definitely true, but so far unproved) can be settled. $g(6) = 73$ is due to Pillai and $g(5) = 37$ was proved by Chen Jing Run. $g(3) = 9$ is known from around 1910, leaving only $g(4)$ in doubt.

Recently, the author, J. M. Deshouillers and F. Dress proved that every integer $\geq 10^{367}$ is expressible as a sum of 19 biquadrates and when the computer calculations showed that every integer upto 10^{367} is a sum of 19 biquadrates, the problem was completely settled.

5. *Other problems*: We shall briefly touch upon the other problems, wherein the circle method has played an important role. Since as we observed earlier, small numbers like

$h(k)$ required maximum number of summands, it is conceivable that large numbers might require less number of summands. Accordingly, let us define, following G. H. Hardy, $G(k)$ to be the minimum number of summands required for writing every sufficiently large integer as a sum of k^{th} powers. Even though much progress has been achieved in computing the value of $G(k)$, essentially due to the efforts of Thanigasalam and Vaughan in recent times, the exact value of $G(k)$ is not known except in two cases. The result $G(2) = 4$ is already due to Lagrange and $G(4) = 16$ was proved by Davenport.

The celebrated conjecture of Goldbach states that every even integer is a sum of two prime numbers. Even though the conjecture is not yet proven, some partial results have been proved. Using the circle method, I. M. Vinogradov proved that every sufficiently large odd integer is a sum of three prime numbers. Using the circle method and sieve methods, Chen Jing Run proved that every sufficiently large even integer $2n$ can be written as $a+b$, where a can be taken to be a prime and b having at most two prime factors.

Look at the following combinatorial problem suggested by P. Erdös. Let A be a subset of integers of $\{n : 1 \leq n \leq N\}$. Assume that no k elements of A are in arithmetic progression. Then, Erdös conjectured that the number of elements of A is $o(N)$.

$$\text{That is, as } N \to \infty, \ \frac{\text{Number of integers in } A}{N} \to 0$$

The case $k = 3$ was first settled by K. F. Roth essentially by the circle method. Later, the general case was solved by E. Szemeredi by combinatorial methods, and Furstenberg by ergodic methods. It is pertinent to point out that the methods adopted by Szemeredi and Furstenberg, give only $o(N)$, whereas Roth's method gives a slightly better estimate for the number of elements of A.

The potential of the circle method has not yet been completely exhausted and the Ramanujan-Hardy-Littlewood circle method remains one of the most powerful tools in the hands of analytic number theorists.

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Impact of Ramanujan's work on modern mathematics

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Since the title of this talk seems somewhat sweeping and perhaps a little too ambitious as well, we can only try our best to do justice to the same, subject to various constraints.

First, let us mention, in passing, a tiny anecdote attributed to B. M. Wilson describing, in some sense, the impact of Ramanujan's work on an eminent mathematician (rather than on mathematics!). When Professor G. Polya was visiting Oxford some time in 1925, he borrowed from Professor G. H. Hardy, his copy of the *Notebooks of Srinivasa Ramanujan*. A couple of days later, he returned them to Professor Hardy, virtually in a state of panic, with the explanation that however long he might manage to retain the *Notebooks*, he would have to keep attempting to verify the formulae stated therein and would never again have the time to prove another original result of his own!

We begin now with a celebrated (and the most famous) conjecture of Ramanujan subsumed under the Ramanujan–Petersson conjecture on the order of magnitude of Fourier coefficients of cusp forms (of integral weight); as we proceed, the reader can well visualize its impact (as well as that of Ramanujan's entire work) on present-day mathematics and the fillip provided thereby for rapid mathematical advances. Ramanujan's conjecture concerned the estimation of the function $\tau(n)$ relative to n, where $\tau(n)$ is the coefficient of x^n in the power-series expansion of the function $f(x)$: $x\prod_{m=1}^{\infty}(1-x^m)^{24}$ for $|x|<1$. By very definition, $\tau(n)$ is an integer which is never zero, according to a conjecture of D. H. Lehmer still open but confirmed for $n \leq 10^{15}$ in view of Serre's results. For complex z with positive imaginary part, $\Delta(z) := f(e^{2\pi i z})$ is the simplest example of a non-trivial cusp form (and indeed of weight 12). A cusp form of weight k (for the modular group) is a holomorphic function g of a variable z in the complex upper half-plane with the Fourier expansion $g(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$, such that $g(-1/z) = z^k g(z)$. Such a form g is called a Hecke eigenform, if it is an eigenfunction of all the Hecke operators. In his well-known paper entitled "On certain arithmetical functions", Ramanujan stated multiplicative properties for $\tau(n)$ which together are equivalent to the representability of the Dirichlet series $\sum_{n=1}^{\infty} \tau(n) n^{-s}$

(with complex s having real part large enough) by the Euler product $\prod_p (1-\tau(p)p^{-s} +p^{11-2s})^{-1}$ extended over all the primes $p = 2, 3, 5, 7, \ldots$. Further, he has stated therein that ("as appears to be highly probable") $\{\tau(p)/2\}^2 \leq p^{11}$ for prime p and more generally $\tau(n) \leq n^{11/2}\, d(n)$ for any natural number n, where $d(n)$ is the number of positive integers t dividing n. This is the conjecture made by Ramanujan in 1916 which kept at bay a whole galaxy of distinguished mathematicians for nearly six decades until its validity was established in 1974 by the outstanding French mathematician P. Deligne, using very sophisticated and powerful techniques from algebraic geometry. The first assertion of Ramanujan concerning the representation by an Euler product was verified by L. J. Mordell and this was a remarkable forerunner of Hecke theory to emerge 20 years later. We shall revert to this aspect of Ramanujan's work in a little while and first elaborate somewhat on the confirmation of his conjecture on the order of magnitude of $\tau(n)$. By a sophisticated method, Ramanujan himself had shown that $|\tau(n)|/n^7$ is bounded (independently of n) or $\tau(n) = O(n^7)$ in the O-notation of Hardy–Littlewood. Hardy improved this to the estimate $\tau(n) = O(n^6)$, with a function-theoretic method. By an elegant technique (now known as Rankin's "convolution trick" or the "Rankin–Selberg method") for studying the (meromorphic) continuation of the Dirichlet series $\sum_{n=1}^{\infty} \tau^2(n)\, n^{-s}$ as a function of the complex variable s, Rankin could obtain in 1939 the better estimate $\tau(n) = O(n^{6-1/5+\varepsilon})$ for every $\varepsilon > 0$. Appealing to A. Weil's famous result on the Riemann hypothesis for zeta functions of curves, Eichler settled in 1954, the Ramanujan–Petersson conjecture for coefficients of cusp forms of weight 2 for certain congruence subgroups of the modular group. Let us now try to formulate quickly the Riemann hypothesis for the zeta functions above. For any algebraic curve X over a finite field $\mathbb{F}_q$ of q elements, there exists an associated zeta function $Z(X/\mathbb{F}_q, T) = \exp\left(\sum_{m=1}^{\infty} N_m\, T^m/m\right)$ where N_m is the number of points of X "with coordinates" in $\mathbb{F}_{q^m}$, for $m \geq 1$. This zeta function turns out to be of the form $P(T)/\{(1-T)\,(1-qT)\}$ with a polynomial $P(T)$ in the variable T of degree equal to twice the genus g of the curve X. The (corresponding) Riemann hypothesis asserts that, all the complex numbers α_j in $P(T) = \prod_{j=1}^{2g} (1-\alpha_j T)$, have absolute value q. Not only did A. Weil establish the truth of the Riemann hypothesis for zeta functions of curves X over finite fields but he also formulated similar conjectures for varieties (instead of curves) over finite fields. Based on his application of this Riemann hypothesis to the estimation of exponential sums, Weil suggested that the confirmation of the Ramanujan conjecture could be reduced to the estimation of the eigenvalues of the Frobenius endomorphism operating on a suitable cohomology group. The search for a "cohomological link" led J. P. Serre to conjecture the existence of a "compatible system of l-adic representations ρ_l" of the Galois group $G = \mathrm{Gal}(K_l/\mathbb{Q})$ of the maximal galois extension K_l of the field $\mathbb{Q}$ of rational numbers which is unramified over $\mathbb{Q}$ except at a given prime l, on a 2-dimensional vector space V over the field $\mathbb{Q}_l$ of l-adic numbers, such that for any prime number p different from l, the Hecke polynomial $1-\tau(p)T+p^{11}T^2$ is the characteristic polynomial of the "Frobenius automorphism" F_p in G. Indeed, Deligne proved the extremely important result that for any Hecke eigenform φ of weight k with the Fourier expansion $\varphi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and the given prime l, there exists an l-adic representation $\rho_l = \rho_l(f)$ of the Galois group of the algebraic closure of $\mathbb{Q}$ on V such that for prime $p \neq l$, the characteristic polynomial of the image under ρ_l of the Frobenius automorphism corresponding to p is precisely the Hecke polynomial $1-a_p T+p^{k-1} T^2$. Deligne showed further that for the Fourier coefficient a_n of a cusp form of (integral) weight k for a congruence subgroup (of the modular group), the estimate $a_n = O(n^{(k-1)/2+\varepsilon})$ and in particular, Ramanujan's estimate for $\tau(n)$ could be derived as a consequence of the Weil conjectures (for varieties). Now, if, for a given prime p, there existed an n-dimensional smooth projective variety X over the field $\mathbb{F}_p$ (of p elements) such that the Hecke polynomial $1-\tau(p)T+p^{11}T^2 = (1-\alpha(p)T)(1-\overline{\alpha(p)}T)$, with complex conjugates $\alpha(p), \overline{\alpha(p)}$, divided the polynomial $P_{11}(X/\mathbb{F}_p, T) = P_{11}(T)$ featuring in the zeta function $Z(X/\mathbb{F}_p, T) = \dfrac{P_1(T)P_3(T)\ldots P_{2n-1}(T)}{P_0(T)P_2(T)\ldots P_{2n}(T)}$, then by the Weil conjectures for X, all the complex numbers $\alpha_{j,r}$ in $P_j(T) = \prod_r (1-\alpha_{j,r}T)$ will have absolute value $p^{j/2}$; so $|\alpha(p)| = p^{11/2}$, implying that $\tau(p) = \alpha(p)+\overline{\alpha(p)} \leq 2p^{11/2}$. Eichler who took the first step in the direction of the "cohomological link" for the Ramanujan conjecture was followed by Shimura and others in the quest for an X which "should have worked" but their X "was neither compact nor had a smooth compactification". Deligne overcame the difficulties by combining "Lefschetz monodromy techniques" with a subtle generalization of Rankin's "convolution trick" and realizing the l-adic representations ρ_l as "a piece" of the appropriate "l-adic cohomology" in a "smooth variety". The interested reader is referred to an excellent survey of Deligne's work given by N. M. Katz[1].

It was not just for the Dirichlet series associated with the cusp form $\Delta(z)$ of weight 12 that Ramanujan wrote down the Euler product but indeed systematically for those attached to a whole lot of other modular forms f (for congruence subgroups) formed from Dedekind's η-function and normalized Eisenstein series E_4, E_6 respectively of weights 4, 6 etc. It is even more fantastic that when Dirichlet series corresponding to some of these f failed to have an Euler product, he wrote down appropriate linear combinations of such f so that the Euler product property could be restored. In other words, Ramanujan not only anticipated Hecke's discovery (20 years later) that eigenfunctions of the algebra of Hecke operators correspond to Dirichlet series with Euler products but even proceeded to "diagonalize Hecke operators" on specific spaces generated by modular forms f as above. In the same manuscript highlighted by B. J. Birch wherein Ramanujan has pointed out Euler product expansions for Dirichlet series arising from various modular forms, one finds also a list of 40 identities involving modular functions for congruence subgroups (or functions of the type of G, H defined below). The first of these forty identities is connected with the "modular equation of degree 5". Just to give a flavour of these identities, let us quote one (proved by G. N. Watson): $H(q)\,G(q^{11})-q^2\,G(q)\,H(q^{11}) = 1$ for $|q| < 1$, with $G(q) := 1\Big/\left\{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})\right\}$, $H(q) := 1\Big/\left\{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})\right\}$. We also note that the celebrated Rogers–Ramanujan identities (providing a surprising relationship between certain infinite products and some infinite series) are just

$$G(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$H(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

As stated by Watson who proved several of these forty identities, their beauty is comparable with that of the Rogers–Ramanujan identities. L. J. Rogers to whom Ramanujan had communicated them, published proofs of nine of them. Ramanujan's own proofs for the identities are not known. Thanks to the recent work of D. M. Bressoud and A. J. Biagioli, the proofs of all the forty identities are at hand. Most proofs are combinatorial in nature but one should mention that some of them admit of interesting interpretations *via* "modular units". A great deal of work generalizing the Rogers–Ramanujan identities has been carried out by G. N. Watson, W. N. Bailey, B. Gordon, L. J. Rogers, L. J. Slater and others and most notably by G. E. Andrews who has imbedded the classical identities of this type in an infinite family of multiple series identities. Rogers–Ramanujan identities feature also in statistical mechanics in the calculation of sublattice densities of the hard hexagon model, witness the recent important work of R. J. Baxter, G. E. Andrews, etc. who greatly generalize the Rogers–Ramanujan and similar identities in the study of the eight-vertex solid-on-solid model.

Ramanujan has discovered several identities involving modular forms of the same type, in the context of his famous congruences for the partition function, for example. We would like to remark here that with his extremely powerful intuition, Ramanujan seems to have been influenced by the fact that a given modular form cannot afford to have zeros of very large order without vanishing identically and hence to verify the equality of two modular forms of the same type, it suffices to check the equality of their corresponding first few Fourier coefficients.

If $p(n)$ denotes the number of unrestricted partitions of a natural number n, then $p(n)$ is the coefficient of x^n in $\prod_{m=1}^{\infty}(1-x^m)^{-1}$. The Hardy–Ramanujan formula[2] for $p(n)$ is given by

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{q=1}^{\nu} \sqrt{q} \left(\sum_{\substack{1 \le p \le q \\ (p,q)=1}} w_{p,q}\, e^{-2np\pi i/q} \right) \cdot$$

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{e^{\pi\sqrt{(2/3)}\,\lambda_n/q}}{\lambda_n} \right) + O(n^{-1/4})$$

where ν is the integer nearest to $\alpha\sqrt{n}, \alpha$ is any positive constant, $\lambda_n = \left(n - \frac{1}{24}\right)^{1/2}$, $w_{p,q}$ is a root of unity, (p,q) is the greatest common divisor of p and q and the constants in the O-term depend on α. It is the first asymptotic formula of its kind giving the order of magnitude for $p(n)$ while the earlier development of the theory of partitions was exclusively from an algebraic standpoint. For all large n, $p(n)$ is the integer nearest to the main term in the asymptotic formula. This truly remarkable and resplendent formula emerged, as J. E. Littlewood has stated, from "a singularly happy collaboration of two mathematicians, of quite unlike gifts, in which each contributed the best, most characteristic and most fortunate work that was in him". It represents the glorious triumph of the analytical finesse embodied in the powerful Hardy–Littlewood–Ramanujan circle method, coupled with Ramanujan's profound theoretical insight, for the eventual demolition of the impregnable defences behind which the theorem was entrenched. Ramanujan had noted even in his early years that the term corresponding to $q = 1$ was already a good approximation to $p(n)$. With his tremendous intuition, he seems to have continued to insist, in the course of his collaboration with Hardy, that there should exist a formula for $p(n)$ with 'bounded error' and "this was absolutely the essential and the most important contribution" to the discovery of the Hardy–Ramanujan formula for $p(n)$. The complete result might well have never come about, had it not been vouchsafed by Ramanujan's great insight and unfailing feeling for (form or choice of) the correct (algebraic) expressions (to be sought for therein)–including the derivation $\frac{\mathrm{d}}{\mathrm{d}n}$ or the mysterious-looking $n-1/24$ figuring there–which constituted simply an "extraordinary stroke of formal genius". By a clever adaptation of the method of Hardy and Ramanujan, H. Rademacher obtained in 1937, the equally beautiful convergent series representation for $p(n)$, in lieu of an asymptotic formula, namely

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{q=1}^{\infty} \sqrt{q} \left(\sum_{\substack{1 \le p \le q \\ (p,q)=1}} w_{p,q}\, e^{-2np\, \pi i/q} \right) \frac{\mathrm{d}}{\mathrm{d}n} \left(\frac{\sin h(\pi \sqrt{(2/3)}\, \lambda_n/q)}{\lambda_n} \right)$$

The 'circle method' and its various generalizations continue to provide powerful ammunition in the realm of analytic number theory even today!

Ramanujan discovered congruence properties of $p(n)$ for the moduli given by powers of the primes 5, 7, 11 and gave extremely beautiful 'modular' identities in order to establish the congruences for the moduli 5 or 7; by applying to the identities an averaging operator (which one may now recognize as a familiar object in Hecke theory), he derived the latter congruences. These 'modular' identities together with ones 'allied' to them as given by Ramanujan turn out to be simple (linear) relations between modular forms of weight 2 or 3 for congruence subgroups. The general Ramanujan conjecture $p(n) \equiv 0 \pmod{5^\alpha 7^\beta 11^\gamma}$ for $24n \equiv 1 \pmod{5^\alpha 7^b 11^\gamma}$ and integers $\alpha, b, \gamma \geq 0$ and integral β: $= 1 + b/2$ or $(1+b)/2$, got settled finally in 1967 thanks to the remarkable work of A. O. L. Atkin. Earlier, G. N. Watson had confirmed the conjecture in the case of moduli equal to any power of 5 or some powers of 7. One should also mention that to this interesting area, considerable contributions have been made by many well-known Indian mathematicians.

Beautiful congruences for $\tau(n)$ corresponding to moduli given by powers of 2, 3 and 5 as well the moduli 23 or 691 have been established by Ramanujan: *e.g.* $\tau(n) \equiv \sum_{\mathrm{d}|n} \mathrm{d}^{11}$ (mod 691) where the sum on the right hand side of the congruence is taken over all natural numbers d dividing n. The most remarkable feature of this discovery, however, is that Ramanujan captured precisely all the possible congruences for $\tau(n)$, as confirmed by recent work of J. P. Serre and H. P. F. Swinnerton-Dyer. The latter involves Deligne's theorem on l-adic representations ρ_l associated to Hecke eigenforms, the (related) theory of l-adic modular forms developed by Serre and the analysis of the image under the reduction modulo l of ρ_l in the group $GL_2(\mathbb{Z}/l\mathbb{Z})$. Special congruences (as above) for Fourier coefficients of Hecke eigenforms f were shown to be possible only modulo the corresponding finitely many "exceptional primes" for f.

In furtherance of the congruence relations modulo an integer m which Ramanujan found for the number-theoretic functions $p(n)$ or $\tau(n)$, he also studied related questions of asymptotic distribution corresponding to the residue classes modulo m. For example with $m = 5$, he could show that the number of natural numbers $n \leq x$ with $\tau(n) \not\equiv 0$ (mod 5) is asymptotic to $cx/(\log x)^{1/4}$ as x goes to infinity, with an explicitly determined constant c. One finds in his unpublished manuscripts, analogous results for the moduli 3, 7, 691, etc. (*cf.* papers of G. N. Watson). These results have recently been vastly generalized by J. P. Serre to cover congruence properties, modulo any integer m, of Fourier coefficients of modular forms f of positive integral weight for congruence subgroups and the exponent emerging in lieu of 1/4 above is interpreted anew as the ratio of the cardinality of specified subsets of a suitable Galois group corresponding to f. Ramanujan's assertion (confirmed by Watson) that for 'almost all' numbers n the congruence "$\tau(n) \equiv 0 \pmod{691}$" holds, came within the purview of these elegant results of Serre.

A modular form f, for a congruence subgroup, with the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/d}$ (wherein the Fourier coefficients a_n are integral) is called lacunary if $\lim_{x \to \infty} \frac{1}{x}$ (the number of $n \leq x$ with $a_n \neq 0) = 0$. Analogously, lacunarity modulo any given integer m is defined, just replacing the condition "$a_n \neq 0$" by "a_n not divisible by m". Ramanujan's results on lacunarity (also modulo m) for arithmetical functions are now covered by the interesting work of Serre in recent years. Serre has shown that for the even powers η^{2k} of Dedekind's η-function, lacunarity arises exactly for $k = 1, 2, 3, 4, 5, 7$ or 13; this required the use of l-adic representations attached to η^{2k} in the sense of Deligne.

In the *Notebooks*[3] of Ramanujan, formulae involving infinite series occur with great frequency but one also finds amazing results dealing with integrals and continued fractions. The number-theoretic concentration is noticed in some early chapters of the *Notebooks* and at the end of the *Notebook II* as well as in *Notebook III*, although Ramanujan is primarily referred to as a great number-theorist. A fair portion of Ramanujan's earlier work in India (and, in particular, the results in the *Notebooks*) concerns his rediscovering important classical results (*e.g.* one finds the functional equation for the Riemann zeta function or L-series quietly written down by him!) "pitting his brains against the accumulated wisdom of Europe"; still the *Notebooks* contain a very large number of new theorems (some of which have been proved independently by others while many are still perhaps awaiting closer investigation). Ramanujan secured for himself Jacobi's formula for the number of ways of representing an integer as a sum of 4 squares of integers, also the analogues for the case of 6 or 8 squares and Legendre's theorem for sums of 3 squares. That Ramanujan discovered for himself the form of the prime number theorem (and "virtually" had the Hardy–Littlewood proof for the same) was an extremely significant achievement putting him in the same class as very great mathematicians like Legendre, Gauss and Dirichlet who had also found the form of this theorem. Ramanujan's result that "almost all numbers have log log n prime factors" has still an air of freshness about it. His remarkable work on highly composite numbers is very important, being pursued by eminent contemporary mathematicians, although the main problem in this direction has not been completely resolved. In classical algebra, Ramanujan's work noted for its great virtuosity was on hypergeometric series (wherein he discovered much of the earlier work contained, for example, in W. N. Bailey's tract) and on continued fractions. He "could only be compared with Euler or Jacobi", with his amazing insight into algebraical formulae, transformations of infinite series, etc. Concerning his work on hypergeometric series, Hardy wrote a paper, devoted to an entire chapter of the *Notebooks*. Ramanujan's masterpiece on continued fractions was his work on the celebrated Rogers–Ramanujan continued fraction

$$C(q) := 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots = \frac{G(q)}{H(q)}$$

with G, H defined earlier and it is related to the Rogers–Ramanujan identities. In his very first letter to Hardy written from India, Ramanujan had included many corollaries of this result by which Hardy confesses to have been "defeated completely". Ramanujan's *Lost Notebook*[4] consisting of about 140 unpaginated sheets unearthed by Professor G. E. Andrews in 1974 from the estate of the late Professor G. N. Watson–contains over 600 identities and broadly speaking, deals with continued fractions of the Rogers–Ramanujan type, q-series and related theta functions (including mock theta functions referred to by Ramanujan in his letter of 12 January 1920 to Hardy and false theta functions). Identities expressing $C(q)$, $(C(q))^2$, $1/C(q)$, $1/(C(q))^3$, etc. in terms of generalized Lambert series besides an 'outrageous' integral representation for $C(q)$ manifesting Ramanujan's "flair for funding the most outstanding version of any particular identity" are to be found in this invaluable manuscript. Ramanujan has also stated therein several results involving such continued fractions and connected with modular equations. In recent years, many of these results have been studied exhaustively and extensively by G. E. Andrews, R. A. Askey, M. Hirschhorn and others. There exist now multilateral summation theorems in higher dimensions due to R. A. Gustafson and S. C. Milne, generalizing Ramanujan's ${}_1\psi_1$ summation with well-known applications to the treatment of orthogonal polynomials defined by basic hypergeometric series. One finds in the '*Lost Notebook*' a systematic treatment, by Ramanujan, of elliptic integrals arising from modular curves of low conductor and several beautiful modular identities; the deep mystery possibly lurking behind them ought to be unravelled.

Ramanujan's elegant paper on integral diagonal quaternary quadratic forms representing integrally all natural numbers (or possibly with finitely many exceptions) reveals his mastery in the arithmetic theory (of quadratic forms). Ramanujan's conjecture that the diophantine equation $x^2+7 = 2^{n+2}$ has only five solutions in pairs (x, n) of natural numbers brought forth in its trail much interesting work from many mathematicians, was first proved by T. Nagell in 1960 and even generalized later by H. Hasse. A clever method of Ramanujan's, from a seemingly little known paper of his in 1912 making use of the method of partial fractions to solve for indeterminates m_j, x_j in the system of $2t$ equations (with given constants s_r): $\sum_{j=1}^{2t} m_j x_j^{r-1} = s_r$ $(1 \le r \le 2t)$, has been applied to obtain a decoding algorithm (for special BCH codes).

Using modular equations of various degrees, Ramanujan calculated the classical invariants in the theory of complex multiplication. With the help of formulae involving modular invariants, he succeeded in getting excellent approximations to the number π or perimeters of ellipses. His undoubtedly deep work on modular functions and singular moduli deserves complete exploration. Interesting work in furtherance of his work on approximations to π has been carried out recently by J. M. Borwein; one should also refer, in this connection, to an important survey by G. Chudnovsky on "Approximations and complex multiplication according to Ramanujan".

There can be no more sublime tribute to Ramanujan nor any more convincing testimony to his colossal mathematical stature than the fact that numerous outstanding mathematicians of the present day have intensively studied his work and continue to pursue it to still farther heights of glory resulting in the blazing of new trails and the blossoming of wholly new branches in mathematics.

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