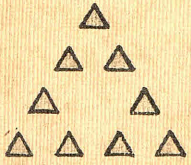

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A MANUAL
OF
GEOMETRY

BY

N. K. NARASIMHA MURTHY, M.Sc.,
Asst. Professor of Mathematics,
Mysore University.

(REVISED EDITION)



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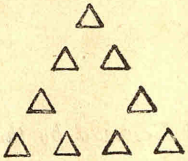
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MOUNT CARMEL COLLEGE
Date.....
BANGALORE.

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प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ।

—*Bhaskara*

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FOREWORD

This Manual of Geometry has been specially prepared to meet the requirements of the students who appear for the Intermediate Examination of the Mysore University. I shall feel amply rewarded if those, for whom the book is intended, find it useful.

Bangalore
July 1942

N. K. NARASIMHA MURTHY

Preface to the Second Edition

In this edition I have introduced alterations and additions, which, I hope will enhance the usefulness of the book to the students.

I am thankful to the authorities of the Mysore University for kindly prescribing this as a text-book.

Bangalore
June 1943

N. K. NARASIMHA MURTHY

Preface to the Fourth Edition

The matter has now been revised and useful changes are made in the Exercises. The problems which have been set for the I.Sc. examinations of the Mysore University have been marked with the symbol (*m*).

Bangalore
July 1952

N. K. NARASIMHA MURTHY

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CHAPTER I

PRELIMINARY IDEAS

In Geometry we come across several entities such as the line, the angle, the area, and the volume, which require to be measured. Such measurement involves the ideas of suitable units and of ratios. For example, to measure the length of a segment of a straight line or a curved line, we must have a standard length such as a foot or a centimetre, which is called a unit. We must then try to find out how many such units are contained in the given line. Again, when we say that a certain angle measures 46°, we mean that 46 angles, each of unit measurement—a degree—can be placed side by side to cover the given angle. It is essential that the student must cultivate the habit of associating the proper unit with each quantity measured. Expressions such as *a line of length 7·2 and a triangle of area 6·5* must be scrupulously avoided. It is however unnecessary to mention the unit when numerical measurements are not involved. Thus "The area of a square whose side is x, is equal to x^2" means that the area of the square is x^2 square-units, the side measuring x units, whatever may be the unit employed.

Two quantities are said to be *commensurable* if they both can be measured in terms of a common unit. Of course they must be of the same kind. If one of them measures l units and the other m units, the ratio of the two quantities is defined as the number l/m and is also written as $l : m$. For example, if we consider two straight lines of length 3·2 cm. and 4·8 cm. a straight line of length 1·6 cm. can be employed to measure the two lines. The first contains two and the second three units. The ratio of their lengths is 2/3 or 2 : 3. By taking a

straight line ·8 cm. long as the unit, the measurements are found to be 4 and 6 and the ratio is 4/6 which is the same as 2 : 3.

A number which cannot be expressed as a fraction p/q, p and q being integers, is called an *irrational number*. Such a number may be represented as the limit of a sequence of numbers or by an approximate decimal number or geometrically by a length. Thus $\sqrt{2}$ is an irrational number which can be represented by the diagonal of a square, each of whose sides is of unit length. It is the limit of the sequences 1, 1·4, 1·41, 1·414, . . . and 2, 1·5, 1·42, 1·415, . . . , the members of the former sequence giving the values of $\sqrt{2}$ to various degrees of approximation. Although many properties enunciated in this book are proved for commensurable magnitudes (for example, Theorems 13, 14, 15, 16 and 28) the results are generalised and applied to incommensurable magnitudes also, the rigorous proofs being beyond the scope of this book. For example, we may say that the ratio of the circumference of a circle to the diameter is constant, although the ratio is incommensurable or irrational.

When the ratio of two quantities is represented as $x : y$, x is called the *antecedent* and y the *consequent*. If $x : y = l : m$, the four quantities whose magnitudes are x, y, l, m are said to be in proportion and x, y, l, m are respectively called the first, the second, the third and the fourth proportionals. If $a : b = b : c$, the middle term b is called the *mean proportional* and c is then the third proportional. Thus we say that four straight lines are in proportion, if the ratio of the first two is equal to that of the next two. We may say that the following are in proportion—(i) a line 4.8 cm. long (ii) a line 7.2 cm. long (iii) a rectangle of area 5·0 sq. in. (iv) a rectangle of

area 7·5 sq. in. But if the second and the third items are interchanged, they are not in proportion for there is no ratio between a straight line and a rectangle, which cannot be measured in terms of a common unit.

If a, b, c, d are in proportion, $a/b=c/d$ or $ad=bc$. Therefore $a/c=b/d$. Also $a/b+1=c/d+1$, which may be written in the form $a+b : b=c+d : d$. This process is called *componendo*. Further $a/b-1=c/d-1$ or $a-b : b=c-d : d$, this process of obtaining the new ratios being called *dividendo*. By division of the two previous sets of ratios, we obtain $a+b : a-b=c+d : c-d$, this process being referred to as *compenendo and dividendo*. In fact, by Algebra, it may be shown that if $a : b=c : d=e : f$....., then each ratio is equal to $(pa+qc+re+\cdots) : (pb+qd+rf+\cdots)$ whatever p, q, r,....may be. This result is called the *Theorem of equal ratios*.

If P be a point on the straight line AB, the line is said to be divided at P. The division is *internal* or *external*, according as P lies between A and B or outside the portion AB of the line. In either case, the parts or *segments* of the line AB are AP and PB. If proper signs are attached to the segments, it is obvious that AP, PB are both of the same sign for internal division and of opposite signs in the case of external division.

If the numbers x, y, z,....are represented by straight lines whose lengths are x, y, z,....units, then their products xy, yz, zx, x^2, y^2, z^2,....can be represented by rectangles and squares whose areas are xy, yz, zx, x^2, y^2, z^2,..... Similarly the products of the third degree xyz, x^2y, z^3,..... can be represented by the rectangular solids (also called *cuboids* or *rectangular parallelopipeds*—for example, a box) whose volumes are of magnitudes xyz, x^2y, z^3,.... The rectangle ABCD, which geometrically represents xy, is denoted by the notation "rect. AC or rect. BD" indicating the opposite vertices and is said to be "the

rectangle contained by AB, BC" indicating the adjacent sides. More briefly we may write "rect. AB. BC" or "rect. ABCD" or even "AB.BC". In a like manner the square ABCD is denoted by "sq. on AB" or more briefly "AB^2". This enables us to express the results of several geometrical theorems in the form of algebraic formulae and vice versa. In this book very often "a line" means a straight line, unless otherwise specially mentioned.

Three or more straight lines which pass through the same point are said to be *concurrent* at that point and three or more points which lie on the same straight line are said to be *collinear*. The student is expected to be familiar with the lines concurrent at the circum-centre, the in-centre, the ex-centres and the centroid of any triangle.

If a point can occupy different positions consistent with given geometrical conditions, the aggregate of all those positions is called the *locus* of the point under the specified conditions. For example, if a point P is always at a distance of 3 cm. from a fixed point O, P lies on the circle whose centre is O and whose radius is 3 cm. The circle is called the locus of P. Any locus must satisfy two conditions-every point satisfying the given conditions must lie on the locus (inclusive aspect) and every point on the locus must satisfy the given conditions (exclusive aspect). Consider, for example, the locus of the point P at which the segment AB of a given line subtends a given angle. P lies on the arc of a circle; but the entire circle does not constitute the locus, for we have to exclude the arc which contains the supplementary angle. Again, the segment of the circle containing the given vertical angle may be described on either side of AB. Therefore the locus of P consists of two equal segments of circles on either side of AB.

CHAPTER II

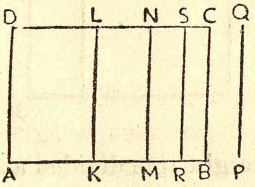
AREAS.

Algebraic Identities.

Theorem 1.

$$x\,(a+b+c+\ldots\ldots)=xa+xb+xc+\ldots\ldots$$

If one of two straight lines is divided into a number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let PQ be the undivided line of magnitude x and AB the other line divided into parts AK, KM, MR,...measuring a, b, c,...respectively. Construct the rectangle ABCD having for its sides AB and BC, the latter being equal to PQ. Draw KL, MN, RS,... parallel to AD, cutting DC in L, N, S,.....

By construction, the figures AC, AL, KN, MS,.....(the quadrilaterals of which the opposite vertices are mentioned) are all rectangles and evidently

$$\text{rect. AC} = \text{rect. AL} + \text{rect. KN} + \text{rect. MS} + \cdots\cdots$$

which proves the theorem.

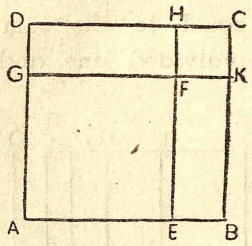
Calculating the areas of the various rectangles, we find that this theorem is equivalent to the algebraic identity,

$$x\,(a+b+c+\ldots\ldots\ldots)=xa+xb+xc+\ldots\ldots\ldots$$

Theorem 2.

$$(a+b)^2 = a^2 + b^2 + 2\,ab$$

If a straight line is divided *internally* into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, *together with* twice the rectangle contained by the parts.



Let AB be a straight line divided at E into two parts AE and EB, measuring respectively a and b units. Describe the square ABCD on AB and draw EH parallel to AD meeting DC in H. Along AD take AG equal to AE and draw GFK parallel to AB cutting EH in F and BC in K.

All the quadrilaterals in the figure are rectangles and the whole figure is made up of four rectangles. Of these, the rectangles AF and FC are squares since AG = AE, and FK = FH = EB. The rectangles EK, GH are of equal area since in each, one of the sides is equal to AE and the other equal to EB.

$\therefore$ sq. on AB = sq. on AE + sq. on EB + 2 rect. AE,EB

or briefly, $AB^2 = AE^2 + EB^2 + 2$ AE.EB.

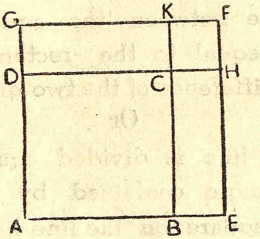
Using algebraic symbols this may be written

$$(a+b)^2 = a^2 + b^2 + 2\,ab.$$

Theorem 3.

$$(a-b)^2 = a^2 + b^2 - 2ab$$

If a straight line is divided *externally* at any point, the square on the line is equal to the sum of the squares on the segments of the line, *diminished by* twice the rectangle contained by the segments.



Let AB be a straight line divided externally at E. Then E is a point in AB produced and the segments of the line are AE and EB, say of lengths a and b units. AB is equal to $a-b$ units. Describe the square ABCD on AB and draw EH parallel to AD so as to meet DC produced in H. Along AD take AG equal to AE and draw GKF parallel to AB cutting EH produced in F and BC produced in K.

All the quadrilaterals in the figure are rectangles.

The rectangle ABCD = rect. AF − rect. BF − rect. DK
= rect. AF − rect. BF − (rect. DF − rect. CF)
= rect. AF + rect. CF − rect. BF. − rect. DF.

But ABCD is a square on the side AB, rect. AF is a square whose side is AE, rect. CF is a square whose side CH or CK is equal to BE and the rectangles BF, DF are equal, their sides being equal to AE and BE.

$\therefore$ sq. on AB = sq. on AE + sq. on BE − 2 rect. AE,BE

or briefly $AB^2 = AE^2 + BE^2 - 2\ AE.BE$

or $(a-b)^2 = a^2 + b^2 - 2ab$.

Note. If the segments are taken with proper signs, the result of this theorem as well as that of the previous theorem may be written in the same form

$$AB^2 = AE^2 + EB^2 + 2\ AE.EB,$$

E being any point in AB or AB produced.

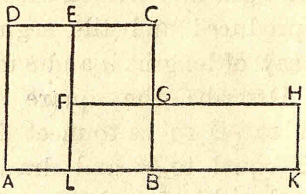
Theorem 4.

$$a^2 - b^2 = (a+b)\ (a-b)$$

The difference between the squares on two given straight lines is equal to the rectangle contained by the sum and the difference of the two given lines.

Or

If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.



Let AB and LB be the two lines. Describe the square ABCD on AB and draw LE parallel to AD cutting DC at E. In LE take LF equal to AL and produce AB to K making BK = AB. Complete the parallelogram FLKH and let FH cut BC at G.

All the quadrilaterals in the figure are rectangles and FC is a square whose side is equal to LB.

$\therefore$ BK = AB = AD, KH = BG = LF = AL and

rect. DL = rect. GK.

$\therefore$ sq. on AB − sq on LB = rect. AC − rect. FC

= rect. DL + rect. FB = rect. GK + rect. FB = rect. FK,

whose sides are equal to AB + LB and AB − LB.

$$\therefore AB^2 - LB^2 = (AB + LB)(AB - LB)$$
$$\text{or } a^2 - b^2 = (a+b)(a-b)$$

where $a = AB$ and $b = LB$.

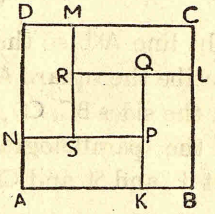
Note. Sq. on AB = rect. FK + sq. on LB and this leads to the second form of enunciation where AK is the line divided equally at B and unequally at L. In symbols this may be written

$$a^2 = (a+b)(a-b) + b^2.$$

Theorem 5.

$$(a+b)^2 - (a-b)^2 = 4\ ab$$

The difference between the squares on the sum and the difference of two straight lines is equal to four times the rectangle contained by the two lines.



Draw AB equal to the larger line and in it mark the point K so that KB is equal to the smaller line. Describe the square ABCD on AB and on its sides BC, CD, DA take the points L, M, N, respectively so that BL = CM = DN = AK. Through K, L, M, N draw parallels to BC, CD, DA, AB as indicated in the diagram. Let $AK = a$ and $KB = b$.

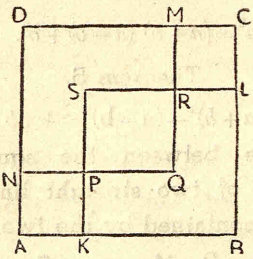
The rectangles AP, BQ, CR and DS are of equal area, their sides being equal to AK and KB. The rectangle PR is a square, since SP = PQ, each being equal to KQ − KP or AK − KB.

$$\therefore \text{sq. } AC - \text{sq. } PR = 4 \text{ rect. } AP = 4 \text{ rect. } AK, KB$$
$$\text{or } (AK + KB)^2 - (AK - KB)^2 = 4\ AK.KB$$
$$\text{or } (a+b)^2 - (a-b)^2 = 4\ ab.$$

Theorem 6.

$$(a+b)^2+(a-b)^2=2a^2+2b^2$$

The sum of the squares on the sum and the difference of two straight lines is equal to twice the sum of the squares on the two lines.



Draw the straight line AKB so that AK, KB are equal to the given lines. Describe the square ABCD on AB and mark the points L, M, N on the sides BC, CD, DA so that LC = CM = NA = AK. Complete the parallelograms KBLS and MDNQ. Let NQ and SK cut at P, and SL and QM cut at R.

The rectangles AP, RC are equal squares whose sides are equal to AK and the rectangles KL, DQ are also equal squares whose sides ars equal to KB.

PR is a square since PQ = QR = QM − RM = KB − AK.

From the diagram we see that

rect. AP + rect. KL + rect. RC + rect. DQ

= rect. AC + rect. PR.

All these rectangles being squares, this may be written

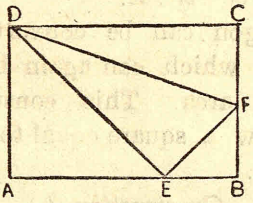
2 (sq. on AK + sq. on KB) = sq. on AB + sq. on PQ

or $(AK+KB)^2+(KB-AK)^2=2(AK^2+KB^2)$.

If KB = a, AK = b, this may be put in the form

$$(a+b)^2+(a-b)^2=2(a^2+b^2).$$

Alternative proof.



Draw the straight line AEB, such that AE,EB are equal to the given lines, AE being the greater. Describe the rectangle ABCD on AB, making AD = BC = AE. In BC mark F so that BF = EB. Join DE, EF, FD.

ADE, BEF are right-angled isosceles triangles.

$\therefore \angle AED = 45^\circ = \angle BEF$. $\therefore \angle DEF = 90^\circ$ and $DF^2 = DE^2 + EF^2$.

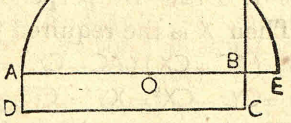
But $DF^2 = DC^2 + CF^2$, $DE^2 = AE^2 + AD^2 = 2AE^2$, $EF^2 = BF^2 + EB^2 = 2EB^2$.

$$\therefore DC^2 + CF^2 = 2AE^2 + 2EB^2$$

or $$(AE + EB)^2 + (AE - EB)^2 = 2AE^2 + 2EB^2.$$

Construction 1.

To describe a square equal to a given rectangle.



Let ABCD be the given rectangle. Produce AB to E making BE equal to BC. Describe the semi-circle on AE as diameter (O being the centre) and erect BF perp. to AE at B so as to cut the circle at F. BF is the side of the required square which can now be completed.

Proof. If $AO = OE = OF = a$ and $OB = b$,

$AB.BC = AB.BE = (a + b)(a - b) = a^2 - b^2 = OF^2 - OB^2 = BF^2$.

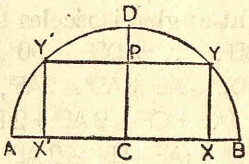
BF is also said to be the *mean proportional* between AB and BE since AB : BF = BF : BE.

Note. A polygon can be converted into a triangle of the same area, which can again be converted into a rectangle of equal area. This construction therefore enables us to draw a square equal to a given rectangle, triangle or polygon.

Construction 2.

To divide a straight line so that the rectangle contained by the segments is equal to a given square.

Case (i) *Internal division.*



Let AB be the given line and C its mid-point. Draw a semi-circle with C as centre and AC as radius. In the radius perpendicular to AB take P so that CP is equal to the side of the given square. Draw PY parallel to AB to cut the circle at Y. Draw YX perpendicular to AB and meeting it in X. Then X is the required point of division.

Proof. $AX.XB = (AC + CX)(AC - CX) = AC^2 - CX^2$

$= CY^2 - CX^2 = XY^2 = CP^2$.

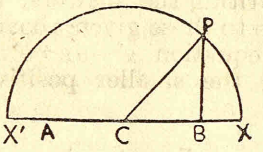
Notes. (1) We also get another point X′ such that X′C = CX, satisfying the given conditions.

(2) The construction becomes impossible if the side of the given square is greater than half the given line (*viz.* CP > AC).

(3) If $AB = a$ and $CP = b$, this construction geometrically solves the equations $x + y = a$, $xy = b^2$.

$x = AX$, $y = XB$, or $x = XB$, $y = AX$.

Case (ii) *External division.*



Let AB be the given line and C its mid-point. Erect BP perpendicular to it and equal to the side of the given square. With centre C and radius CP draw a circle to cut AB produced at X (or X′). Then X (or X′) is the required point of division.

Proof. $AX.BX = (CX + CB)(CX - CB) = CX^2 - CB^2$
$= CP^2 - CB^2 = BP^2.$

Notes. (1) This construction is always possible.

(2) If $AB = a$, $BP = b$, the solutions of $x - y = a$, $xy = b^2$ are $x = AX$, $y = BX$; or $x = -X'A$, $y = -X'B$.

(3) In the case of internal division, if $AB = a$ and $CP = b$, and AX or $XB = x$, we get in either case

$$AX \cdot XB = CP^2, \text{ or } x(a - x) = b^2.$$
$$\therefore x^2 - ax + b^2 = 0.$$

Hence, to solve geometrically the equation $x^2 - ax + b^2 = 0$, divide internally at X, a straight line AB of length a, so that the rectangle contained by the segments is equal to the square on the side equal to b. Then AX, XB are the two roots, both being taken with the positive sign.

(4) In the case of external division, if $AB = a$, $BP = b$ and AX (or $-BX$) $= x$, we get $AX.BX = BP^2$, which is equivalent to

$$x(x - a) = b^2 \quad \text{or} \quad x^2 - ax - b^2 = 0.$$

Hence, to solve geometrically the equation $x^2 - ax - b^2 = 0$, divide externally a straight line of length a, so that the rectangle contained by the segments is equal to the square on the side equal to b. The larger segment with the positive sign and the smaller with the negative sign are the roots.

(5) If a is negative, the same methods are employed but in writing the answers, the roots are taken with signs opposite to those given above, *viz.*, both roots negative for the equation $x^2 - ax + b^2 = 0$ and the larger root negative and the smaller positive for the equation $x^2 - ax - b^2 = 0$.

Exercise 1

(1) Illustrate by diagrams the identities

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca,$

$(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca.$

(2) If a straight line AB is bisected at X and if Y is any other point on the line, prove that

(i) $AY - YB = 2XY$, $\quad AY + YB = AB = 2AX$,

(ii) $AY^2 - YB^2 = 2AB.XY$, $\quad AY^2 + YB^2 = 2(AX^2 + XY^2)$

(iii) $AY.YB = AX^2 - XY^2$.

(3) If A, B, C, D are four consecutive points on a straight line, show that $AC.BD = AB.CD + AD.BC$.

(4) AD is the perpendicular from A to the hypotenuse BC of a right-angled triangle ABC. Show that (i) $AD^2 = BD.DC$ (ii) $AB^2 = BD.BC$ (iii) $CA^2 = CD.CB$.

(5) Show how to find a point X in a given straight line AB such that (i) the sum or (ii) the difference of the squares on AX and XB may be equal to a given square.

(6) Evaluate geometrically $\sqrt{3.4 \times 4.5}$, $\sqrt{1.2}$, 2.3^2.

(7) Draw a straight line AB equal to 2.6″ and divide it internally at X and externally at Y, so that in each case the rectangle contained by the segments is equal to 1.21 square inches.

(8) Solve geometrically (i) $x^2 - 4.8x + 3.6 = 0$,

(ii) $x^2 - 1.5x - 10 = 0$, (iii) $x^2 + 4x - 12 = 0$,

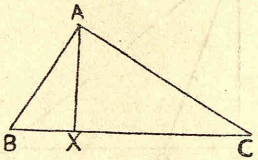
(iv) $x + y = 5$, $xy = 4$, (v) $x - y = 2.5$, $xy = 9$.

Date .....

2. Extension of the Theorem of Pythagoras

Theorem 7.

The square on the side opposite to an *acute* angle of any triangle is equal to the sum of the squares on the remaining sides *diminished by* twice the rectangle contained by one of those sides and the projection of the other upon it.



Let ABC be a triangle in which B is an acute angle and let AX be the perpendicular from A to BC. BX is the projection of AB on BC. It is required to prove that

$$AC^2 = AB^2 + BC^2 - 2\ BC.BX.$$

$AC^2 = AX^2 + XC^2$, (AXC being a right angle)

$= AX^2 + (BC - BX)^2$

$= AX^2 + BC^2 + BX^2 - 2\ BC.BX$

$= AB^2 + BC^2 - 2\ BC.BX.$ (AXB being a right angle)

Notes. (1) If X coincides with C, the result holds good, AC^2 being equal to $AB^2 - BC^2$ and no special proof is necessary.

(2) If X lies outside BC, $CX = BX - BC$ and the rest of the argument remains unaltered.

(3) $BX \div AB = \cos B$. $\therefore BX = AB \cos B$ and the theorem may also be stated thus:

$$AC^2 = AB^2 + BC^2 - 2\ BC.AB \cos B,$$

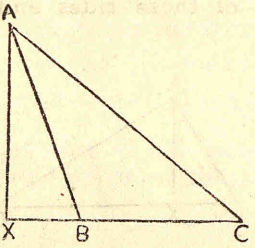
or using small letters to denote the sides,

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

Similar results hold good for the squares on other sides.

Theorem 8.

In an obtuse-angled triangle, the square on the side opposite the *obtuse* angle is equal to the sum of the squares on the remaining sides *together with* twice the rectangle contained by one of those sides and the projection of the other upon it.



Let ABC be a triangle in which B is an obtuse angle and AX the perpendicular from A to BC. Then XB is the projection of AB on BC. It is required to prove that

$$AC^2 = AB^2 + BC^2 + 2\ BC \cdot XB.$$

$$\begin{aligned} AC^2 &= AX^2 + XC^2 \\ &= AX^2 + (XB + BC)^2 \\ &= AX^2 + XB^2 + BC^2 + 2\ XB \cdot BC \\ &= AB^2 + BC^2 + 2\ XB \cdot BC. \end{aligned}$$

Notes. (1) If X coincides with B in the figure, $AC^2 = AB^2 + BC^2$. In other words, as the obtuse angle B diminishes and tends to become a right angle, twice the rectangle added to the sum of the squares on the other sides tends to zero.

(2) $$XB \div AB = \cos(180^\circ - B) = -\cos B.$$
$$\therefore XB = -AB \cos B.$$

The theorem may also be stated in the form

$$AC^2 = AB^2 + BC^2 - 2\ AB.BC \cos B$$

or using small letters to denote the sides,

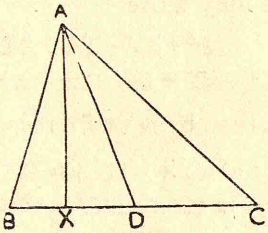
$$b^2 = c^2 + a^2 - 2\ ca \cos B.$$

(3) This *trigonometrical* form is the common expression for the result, whether the angle be acute, right or obtuse. It is called the **cosine formula.**

3. Apollonius' Theorem.

Theorem 9.

The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median bisecting that side.



Let ABC be a triangle in which AD is the median and AX the altitude through A. Let $\angle ADB$ be acute.

ADB is an acute angle of the triangle ABD.

$$\therefore AB^2 = AD^2 + BD^2 - 2\ BD.XD.$$

ADC is an obtuse angle of the triangle ADC.

$$\therefore AC^2 = AD^2 + DC^2 + 2\ DC.XD.$$

$\because BD = DC$, we get by adding,

$$AB^2 + AC^2 = 2\ AD^2 + 2\ BD^2.$$

Notes. (1) If X coincides with D, $\angle ADB = \angle ADC = 90°$ and we easily obtain

$$AB^2 + AC^2 = (AD^2 + BD^2) + (AD^2 + DC^2) = 2\ AD^2 + 2\ BD^2.$$

See also Exercise 1 (2)

(2) The result may also be written in the form $AB^2 + AC^2 = 2\ AD^2 + \frac{1}{2}\ BC^2$, since $BD = \frac{1}{2}\ BC$. This leads to the formula $AD^2 = \frac{1}{2}\ (b^2 + c^2 - \frac{1}{2}\ a^2)$ from which the lengths of the medians of a triangle whose sides are given, can be calculated.

(3) This is called **Apollonius' Theorem.** The result can be generalised as in the next theorem.

Theorem 10.

If D be a point in the side BC of a triangle ABC such m BD $= n$ DC, then

$$m\,AB^2 + n\,AC^2 = (m+n)\,AD^2 + m\,BD^2 + n\,DC^2.$$

Proof. Using the figure and the results of the previous theorem we may write

$$AB^2 = AD^2 + BD^2 - 2\,BD.XD,$$
$$AC^2 = AD^2 + DC^2 + 2\,DC.XD.$$

Multiplying these by m and n respectively and adding,

$$m\,AB^2 + n\,AC^2 = (m+n)\,AD^2 + m\,BD^2 + n\,DC^2 - 2\,XD(m\,BD - n\,DC)$$
$$= (m+n)\,AD^2 + m\,BD^2 + n\,DC^2, \quad \text{since } m\,BD = n\,DC.$$

Notes. (1) $BD = \dfrac{n}{m+n}BC, \quad DC = \dfrac{m}{m+n}BC.$

$$\therefore m\,AB^2 + n\,AC^2 = (m+n)\,AD^2 + m\left(\frac{n}{m+n}\right)^2 BC^2 + n\left(\frac{m}{m+n}\right)^2 BC^2$$
$$= (m+n)\,AD^2 + \frac{mn}{m+n}BC^2$$

(2) If $BD = \dfrac{1}{k}BC$ and $\therefore DC = \left(1 - \dfrac{1}{k}\right)BC,$

then $(k-1)\,BD = DC.$

$$\therefore (k-1)\,AB^2 + AC^2 = k\,AD^2 + (k-1)\,BD^2 + DC^2$$
$$= k\,AD^2 + (k-1)\frac{BC^2}{k^2} + \left(\frac{k-1}{k}\right)^2 BC^2$$
$$= k\,AD^2 + \frac{k-1}{k^2}BC^2(1 + k - 1)$$
$$= k\,AD^2 + \left(1 - \frac{1}{k}\right)BC^2,$$

which gives an alternative form of the result.

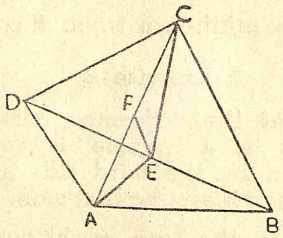
Examples.

(1) **Find the locus of the vertex A of a triangle ABC, given BC and $AB^2 + AC^2$.**

Since $AD^2 = \frac{1}{2}(AB^2 + AC^2 - \frac{1}{2}BC^2)$, AD is known and D is the mid-point of BC. The locus of A is, therefore, the circle whose centre is D and whose radius is equal to $\sqrt{[\frac{1}{2}(AB^2 + AC^2 - \frac{1}{2}BC^2)]}$.

The triangle can be constructed when one other item, such as the altitude through A or a base angle, is known.

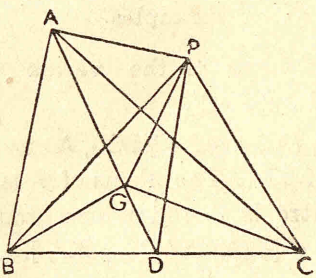
(2) *The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the straight line joining the mid-points of the diagonals.*



Let ABCD be the quadrilateral and E and F the mid-points of the diagonals BD and AC. Join EA, EF, EC.

$$AB^2 + AD^2 = 2AE^2 + \tfrac{1}{2}BD^2 \quad \text{from } \triangle ABD.$$
$$BC^2 + DC^2 = 2CE^2 + \tfrac{1}{2}BD^2, \quad \text{from } \triangle BDC.$$
$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = 2(AE^2 + CE^2) + BD^2$$
$$= 4EF^2 + AC^2 + BD^2, \quad \text{from } \triangle AEC.$$

(3) **If G is the centroid of a triangle ABC and P any other point, show that $PA^2 + PB^2 + PC^2 = 3PG^2 + GA^2 + GB^2 + GC^2$. Find when $PA^2 + PB^2 + PC^2$ is a minimum.**



Let D be the mid-point of BC.

From $\triangle$ PBC, $PB^2 + PC^2 = 2\,PD^2 + 2\,BD^2$.

Since $2\,GD = AG$, we get from $\triangle$ PAD,

$$PA^2 + 2\,PD^2 = (2+1)PG^2 + AG^2 + 2\,GD^2. \qquad \text{(Th. 10)}$$

$\therefore PA^2 + PB^2 + PC^2 = PA^2 + 2\,PD^2 + 2\,BD^2$

$= 3\,PG^2 + AG^2 + 2\,GD^2 + 2\,BD^2 \quad = 3\,PG^2 + AG^2 + BG^2 + CG^2$,

since $BG^2 + CG^2 = 2\,BD^2$, from $\triangle$ GBC.

[Show that this is also equal to $3\,PG^2 + \frac{1}{3}(AB^2 + BC^2 + CA^2)$]

$PA^2 + PB^2 + PC^2$ is minimum when P coincides with G.

Exercise 2

(1) Show that the difference between the squares on any two sides of a triangle is equal to twice the rectangle contained by the third side and the projection upon it of the median bisecting that side.

(2) Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

(3) In a triangle ABC, the angles B and C are acute. If BE, CF are drawn perpendicular to AC, AB respectively, show that $BC^2 = AB.FB + AC.EC$.

(4) Show that four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides.

(5) P is a point on the circumcircle of an equilateral triangle ABC. show that $PA^2 + PB^2 + PC^2 = 2\,BC^2 = 6\,R^2$, R being the circum-radius. (*m*)

(6) The medians of a triangle are 1.3″, 1.9″, 2.2″ long. Show that the triangle is right-angled.

(7) Construct a triangle ABC given $BC = 5$ cm., $AB^2 + AC^2 = 44{\cdot}5$ sq. cm. and

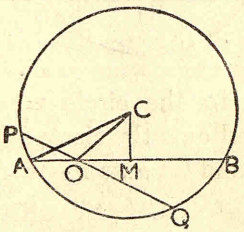
(a) $A = 55°$ or (b) altitude $AX = 3$ cm.

(Find the maximum value of AX which makes the construction possible.)

4. SEGMENTS OF A CHORD.

Theorem 11.

The rectangle contained by the segments of any chord passing through a given point within a circle is constant.



Let AB be a chord of the circle whose centre is C and O the given point through which AB passes. Let M be the mid-point of AB. Then CM is perpendicular to AB.

$$AO.OB = (AM - OM)(MB + OM) = (AM - OM)(AM + OM)$$
$$= AM^2 - OM^2 = (AC^2 - CM^2) - (OC^2 - CM^2)$$
$$= AC^2 - OC^2$$

which is constant since AC is a radius and C and O are fixed points.

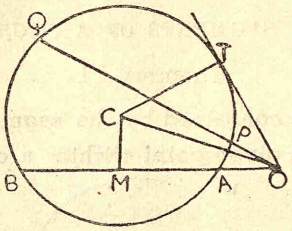
Notes. (1) If AB and PQ are any two chords intersecting at O, then $AO.OB = PO.OQ$.

(2) *Conversely*, if two st. lines AB and PQ intersect at O, such that $AO.OB = PO.OQ$, than A,Q,B,P are concyclic.

For, if the circle through A, B, C cuts PQ again at R, then $PO.OR = AO.OB = PO.OQ$ and therefore $OR = OQ$; that is, R coincides with Q, or Q lies on the circle through A, B, P.

Theorem 12.

If from a point outside a circle any secant is drawn to cut the circle, the rectangle contained by the segments is constant and equal to the square on the tangent from the point to the circle.



OAB is a secant to the circle whose centre is C. It passes through O and cuts the circle at A and B. Let M be the mid-point of AB and OT a tangent from O to the circle.

$$\begin{aligned} OA.OB &= (OM - AM)(OM + MB) = (OM - AM)(OM + AM) \\ &= OM^2 - AM^2 = (OC^2 - CM^2) - (AC^2 - CM^2) \\ &= OC^2 - AC^2 \end{aligned}$$

which is constant since O and C are fixed points and AC is a radius.

Again $OA.OB = OC^2 - AC^2 = OC^2 - CT^2 = OT^2$.

Notes. (1) Whether O is within or without the circle, the rectangle contained by the segments of any chord through it, is equal to the difference between the squares on the radius and on the distance between the point and the centre. It is also equal to the square on the tangent, real or imaginary, from the point and is called the *power of the point*, the signs of the segments being considered.

(2) If OPQ be another secant cutting the circle in P and Q, then $OA.OB = OP.OQ$. In the case of the tangent which meets the circle in coincident points T, T this result becomes $OA.OB = OT.OT = OT^2$.

(3) The converse of the theorem is also true and it can be proved as in the last theorem.

(4) The result can be applied to find the square root of a given number. Thus to determine $\sqrt{2{\cdot}4}$, express 2·4 as the product of two numbers, say 1·2 and 2. Draw a line BAO such that BO = 2, and AO = 1·2 units. Describe any circle to pass through A and B. Draw a tangent OT to the circle from O and measure it. It gives the value of $\sqrt{2{\cdot}4}$, since $OT^2 = OA.OB = 1{\cdot}2 \times 2 = 2{\cdot}4$. In practice it is convenient to draw the circle on AB as diameter.

Exercise 2

(1) Prove that the tangents to two intersecting circles from any point in their common chord produced are equal.

(2) BY and CZ are drawn perpendicular to the opposite sides from the vertices B and C of a triangle ABC. Prove that AY.AC = AZ.AB.

(3) ABC is a triangle right-angled at A. AX is drawn perpendicular to BC to meet it in X. Prove that $AX^2 = BX.XC$ and $AB^2 = BX.BC$. Show that AC touches the circle ABX. Deduce constructions to evaluate geometrically $\sqrt{3{\cdot}6}$, $1{\cdot}3^2$ and $\sqrt{(2+\sqrt{2})}$.

(4) CD is a chord perpendicular to the diameter AB of a circle and intersects it at O. If OA = h, and OC = a, show that the radius of the circle is $(a^2 + h^2)/2h$.

(5) From an external point P a secant PAB is drawn to the circle. PO is perpendicular to a diameter XY, O the foot of the perpendicular being within the circle. Show that $OP^2 = PA.PB + XO.OY$.

What is the result if O lies outside the circle ?

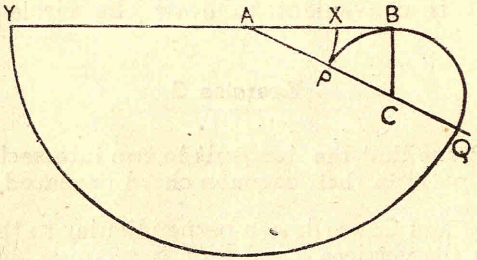
(6) AP, the bisector of the angle A of a triangle ABC is produced to meet the circum-circle at Q. Show that QC is a tangent to the circle APC and hence deduce that

$$BQ^2 = CQ^2 = QP.QA.$$

5. Medial Section

Construction 3.

To divide a straight line into two segments so that the square on one segment may be equal to the rectangle contained by the other segment and the given line.



Let AB be the given line. It is required to divide it at X internally or at Y externally so that

$$AX^2 = AB.XB \quad \text{or} \quad AY^2 = AB.YB.$$

Draw BC perpendicular to AB and equal to half of it. Join CA and in it take a point P so that CP = CB. In AB take X so that AX = AP. Then X divides AB internally in the required manner.

For external division, take Q in AC produced so that CQ = CB and Y in BA produced so that AY = AQ. Then Y is the required point.

(Note that Y is behind A if $AY^2 = AB.YB$.)

Proof. Let $AB = a$ and $AX = AP = x$, so that $BC = PC = \frac{1}{2}a$ and $AC = x + \frac{1}{2}a$.

$\because$ $AC^2 = AB^2 + BC^2$, we have $(x + \frac{1}{2}a)^2 = a^2 + (\frac{1}{2}a)^2$.

$\therefore$ $x^2 + ax = a^2$ or $x^2 = a(a - x)$.

$\therefore$ $AX^2 = AB.XB$.

Notes. (1) The proof for the external division is similar to this and is left as an exercise to the student.

(2) The straight line AB is said to be divided at X or Y in medial section internally or externally. It is also said to be divided in extreme and mean ratio at that point (AX : XB = AB : AX). AX is called the *major* segment and XB the *minor* segment.

(3) From the equation given above,

viz., $x^2 + ax = a^2$, we get $x = \frac{1}{2}(-a \pm \sqrt{5}\,a)$.

$$\therefore\ x/a = (\sqrt{5}-1)/2.$$

The other root $-(\sqrt{5}+1)/2$ gives the ratio of AY to AB, the negative sign indicating that Y is behind A. It can be also directly obtained from the equation $AY^2 = AB.YB$.

(4) Sin $18° = (\sqrt{5}-1)/4$ and hence we can *construct an angle equal to* 18° or its multiples, without a protractor. Draw any line AB and divide it internally in medial section at X. Draw the semi-circle on AB and insert the chord BP in it, equal to half of AX. Then $\angle APB = 90°$ and sin $PAB = BP/AB = \frac{1}{2}\,AX/AB = (\sqrt{5}-1)/4$. $\therefore \angle PAB = 18°$.

(5) If AB is divided at X in internal medial section, then XB is divided at A in external medial section; for in both cases the same relationship holds good, *viz.*, $AX^2 = AB.AB$.

(6) If AB is divided at X internally in medial section and AX is again divided at Z in external medial section, then ZA = AB.

Z A X B

For let $ZA = k$, $AB = a$ and $AX = x$, all being positive. Using the results $AX^2 = AB.XB$ and $AZ^2 = ZX.AX$, we get the equations $x^2 = a(a-x)$ and $k^2 = x(k+x)$, which are equivalent to $a^2 - ax - x^2 = 0$ and $k^2 - kx - x^2 = 0$. $\therefore a/x = k/x = (1+\sqrt{5})/2$, considering the positive values for a, k, x, $\therefore k = a$ or ZA = AB.

(7) If X divides AB internally in medial section, it is possible to determine the position of any one of the three points A, B and X when those of the remaining two are known.

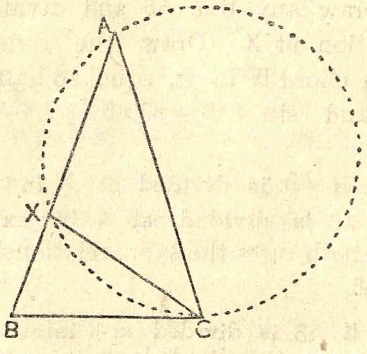
Given A and B, X is determined by Construction 3.

Given X and B, A is determined by dividing XB externally in medial section, as explained in Note (5).

Given A and X, B is determined by dividing AX in external medial section at Z and taking AB equal to ZA, as explained in Note (6).

Construction 4.

To construct an isosceles triangle with each of the base angles double the vertical angle.



Draw any straight line AB and divide it internally in medial section at X. Complete the triangle by taking $BC = AX$ and $AC = AB$.

Proof. $BC^2 = AX^2 = BX \cdot BA$. $\therefore$ BC touches $\odot$ AXC.

$\therefore \angle BCX = \angle XAC$, in the alternate segment.

Adding $\angle ACX$ to both, we get

$\angle XBC = \angle ACB = \angle BCX + \angle ACX = \angle XAC + \angle ACX = \angle BXC$.

$\therefore \angle XBC = \angle BXC$ and $XC = BC = AX$.

$\triangle$ XAC being isosceles, $\angle XAC = \angle ACX$ and also $\angle XAC = \angle BCX$.

$\therefore \angle XAC = \frac{1}{2}(\angle ACX + \angle BCX) = \frac{1}{2} \angle ACB$ or $\frac{1}{2} \angle ABC$.

Notes. (1) The angles of $\triangle$ ABC are, $A = 36°$, $B = C = 72°$.

(2) The angles of $\triangle$ ACX are 36°, 36°, 108°.

(3) Given AB, one of the sides of the triangle ABC, the triangle can be drawn from the above construction. Given the base BC of the same triangle, since BC = AX the major segment of the internal medial section of AB, the length of AB can be determined by dividing BC in external medial section at Y and taking AB = AC = YB.

(4) ACX is an isosceles triangle with one angle three times each of the remaining angles and BCX is equiangular with the triangle ABC. This gives us the method of constructing an isosceles triangle whose vertical angle is equal to three times each of the base angles. The diagram also shows how an isosceles triangle whose base angles are each double the vertical angle can be divided into two isosceles triangles, one of which is equiangular with the given triangle.

(5) If P be the mid-point of BC, $\angle APB = 90°$ and $\angle BAP = 18°$, and the construction given here is substantially the same as that given in Note (4) to Construction 3. This figure enables us to draw angles of 18°, 36°, 72° and 108° without the use of a protractor.

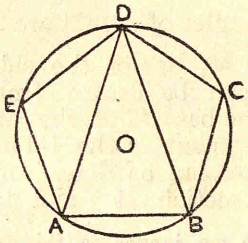
(6) The angle subtended by a side of a regular decagon at the centre O of the circle circumscribing it, is 36°. *We can inscribe a regular decagon or a regular pentagon in a given circle* as follows :—

Draw any radius OA and divide it at X internally in medial section. Starting from A, step off chords AB, BC, CD ... equal to OX all round the circumference. ABCD ... is the required regular decagon. By joining only the alternate vertices or by stepping off chords equal to AC, we get a regular pentagon.

By drawing tangents to the circle at the vertices of the figures drawn, we can describe a regular decagon or a regular pentagon about the circle.

Construction 5.

To construct a regular pentagon or a regular decagon.



Let DA, DB be two diagonals of a regular pentagon ABCDE. Every angle of the pentagon is of magnitude 108°. Therefore from the isosceles triangle BCD we find $\angle DCB = 108°$, $\angle DBC = \angle BDC = 36°$. Similarly $\angle AED = 108°$, $\angle EAD = \angle EDA = 36°$. $\therefore \angle DAB = \angle DBA = 72°$ and $\angle ADB = 36°$.

Thus the central triangle DAB of the figure is an isosceles triangle whose base angles are each double the vertical angle.

Hence *to construct a regular pentagon with a given diagonal,* divide the diagonal in internal medial section and obtain a side (AB is equal to the major segment of the internal medial section of AD). The triangles ABC, ACD and ADE, whose sides are all known, may now be constructed and the figure completed.

If the base be given, draw AB equal to the given base and divide it in external medial section at Y. With centres at A and B draw arcs of circles of radii equal to YA to cut each other at D. With D and B as centres, draw arcs of circles of radii equal to AB cutting at C and with centres at A and D draw arcs of circles of radii equal to AB cutting at E. ABCDE is the required pentagon.

When the difference between a diagonal and a side is given, draw a line equal to it and divide it in external medial section. Of the two segments obtained, the larger one is equal to a diagonal and the smaller one is equal to a side of the pentagon, which can now be constructed. (See Note 5 to Construction 3).

Given the sum of a diagonal and a side, draw a line equal in length to it and divide it internally in medial section. The two segments of the line are equal to the diagonal and the side of the pentagon. Complete the figure as before. (See Note 6 to Construction 3.)

To construct a regular decagon on a given base, draw a line AB equal to the given base and on it describe the triangle OAB so that the verticle angle AOB is equal to half of each of the other angles. Describe the circle with centre O and radius OA and in it insert successively chords BC, CD, DE,.....each equal to AB ; that completes the decagon.

Examples.

(1) *If a straight line* AB *is divided internally in medial section at* X, *show that* $AB^2 + XB^2 = 3\ AX^2$.

A X B

Proof. $AB^2 + XB^2 = (AX + XB)^2 + XB^2$

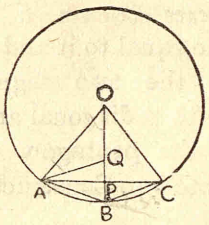
$= AX^2 + 2\ AX.XB + 2\ XB^2$

$= AX^2 + 2\ XB\ (AX + XB)$

$= AX^2 + 2\ XB.AB = AX^2 + 2\ AX^2 = 3AX^2$.

(2) If a regular pentagon, a regular hexagon and a regular decagon are inscribed in a circle and if the lengths of their sides be respectively p, h and d, then

$$p^2 = h^2 + d^2.$$



Let AB and BC be two adjacent sides of the decagon, each equal to d. If O be the centre of the circle, OB bisects AC at right angles at P. In BO take Q so that PQ = BP.

Proof. $\angle AOB = 36^\circ$ and $\angle OAB = \angle ABP = 72^\circ$. $\therefore \angle BAP = 18^\circ$. PAQ, APB are congruent triangles and $\angle BAQ = 36^\circ = \angle OAQ$.

$$\therefore OQ = AQ = AB = d.$$

$\therefore$ Q divides OB in internal medial section.

$\therefore OQ^2 = OB.QB$ or $d^2 = r(r-d)$, r being the radius.

$$\begin{aligned} p^2 = AC^2 &= 4\,AF^2 = 4\,AB^2 - 4\,PB^2 = 4\,AB^2 - QB^2. \\ &= 4d^2 - (r-d)^2 \\ &= 2d^2 + 2r(r-d) - (r-d)^2, \qquad [\because d^2 = r(r-d)] \\ &= d^2 + r^2 = d^2 + h^2. \end{aligned}$$

since the side of the hexagon is equal to the radius.

Notes. (1) This property is the geometrical equivalent of the result $\sin^2 18^\circ + \sin^2 30^\circ = \sin^2 36^\circ$ (applying the formula $a = 2R \sin A$).

(2) $\because d^2 = h(h-d)$,

$$p^2 = d^2 + h^2 = h(h-d) + h^2 = h(2h-d).$$

Exercise 4.

(1) Prove the following construction for medial section:

On the given line AB describe the square ABCD. Let E be the middle point of AD. Produce EA to F so that EF = EB. Complete the square AFGX on AF so that X may lie on AB. Then X divides AB in internal medial section.

(Devise a similar construction for external division.)

(2) If a straight line is divided in internal medial section, the rectangle contained by the sum and the difference of the segments is equal to the rectangle contained by the segments.

(3) If AB is divided internally at X and externally at Y in medial section, prove that

$$1/XB - 1/AX = 1/AB = 1/YB + 1/YA.$$

(4) AB is divided at X internally in medial section. From the larger segment AX, a portion AZ equal to XB is cut off. Show that Z divides AX internally in medial section.

(5) ABC is a triangle in which the angle A is half of B or C. X is a point in AB such that AX = BC. If the bisector of $\angle BAC$ meets the arc XC of the circum-circle of the triangle AXC at P, show that P is the circum-centre of the triangle XBC.

(6) X divides AB in internal medial section. BA is produced to Z so that BZ = 3 BX. Show that $BX^2 = BA.AZ$,

(7) Construct the following figures without using a protractor:

(i) the triangle ABC in which B = C = 2 A, BC = 1·5″.
(ii) the triangle ABC in which B = C = 2 A, AB = 2·7″.
(iii) the triangle ABC in which A = 3 B = 3 C, BC = 8·6 cm,
(iv) the triangle ABC in which A = 3 B = 3 C, AB = 1·8″.
(v) the regular pentagon and the regular decagon inscribed in a circle of radius 1·6″.
(vi) the regular pentagon on a base equal to 5·4 cm.
(vii) the regular pentagon whose diagonals are of length 2·8″.
(viii) the regular pentagon in which the sum of a side and a diagonal = 5·8″.
(ix) the regular pentagon in which the difference between a side and a diagonal = 2·0 cm.
(x) the regular decagon on a base equal to 3·2 cm.
(xi) the regular decagon inscribed in a circle of radius 1·8″.
(xii) the regular pentagon and the regular decagon circumscribed about a circle of radius 3·5 cm.

CHAPTER III.

SIMILAR FIGURES.

1. Ratio and Proportion.

Theorem 13.

A straight line can be divided in a given ratio internally or externally at one and only one point.



Let AB be the st. line to be divided in the ratio $p : q$.

Case (i). For internal division, divide AB into $p+q$ equal parts. If X is the point on AB such that AX contains p parts, then XB contains q parts and $AX : XB = p : q$.

This construction gives a unique point X. For, if there are two points X, X′ both dividing AB in the same ratio, than $AX : XB = AX' : X'B$.

By componendo, $\frac{AX + XB}{XB} = \frac{AX' + X'B}{X'B}$ or $\frac{AB}{XB} = \frac{AB}{X'B}$

$\therefore XB = X'B$ and X′ coincides with X.

Case (ii). For external division, divide AB into $p-q$ equal parts (if $q > p$, take $q-p$ parts). Produce AB (or BA if $q > p$) to Y so that AY contains p such equal parts. Then BY contains q such parts and $AY : YB = p : q$*.

If there are two points Y, Y′ in AB produced such that

$$AY : YB = AY' : Y'B,$$

by dividendo, $\frac{AY - YB}{YB} = \frac{AY' - Y'B}{Y'B}$ or $\frac{AB}{YB} = \frac{AB}{Y'B}$.

$\therefore YB = Y'B$ and Y′ coincides with Y.

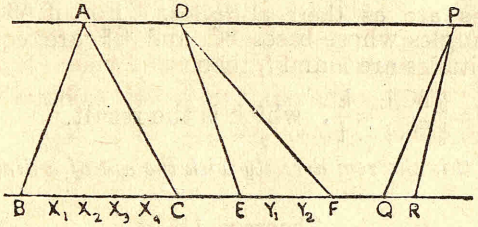
$\therefore$ In either case, there is always one and only one point dividing the given line in the given ratio.

Taking the signs of the segments into account, for external division, $AY : YB = -p : q$, *p and q being positive.*

Note. If the segments are taken with proper signs, one proof applies to both the cases, for in either case, AX + XB = AY + YB = AB. *A straight line cannot be divided externally in the ratio 1 : 1* for AY cannot be equal to YB for any point Y outside AB.

Theorem 14

Areas of triangles having equal altitudes are proportional to their bases.



Let the triangles ABC, DEF having equal altitudes, be placed between the same parallels, as in the diagram. If BC : EF be equal to $p : q$, it is possible to find a length QR so that BC contains p and EF contains q such lengths. Let QR be placed in the line BCEF and let any point P in AD be joined to Q and R.

If BC is divided into p equal parts at $X_1, X_2, X_3, \ldots$ and EF divided into q equal parts at $Y_1, Y_2, Y_3, \ldots$, then

$$BX_1 = X_1X_2 = X_2X_3 = \cdots = EY_1 = Y_1Y_2 = Y_2Y_3 = \cdots = QR.$$

The altitudes of the triangles ABX_1, AX_1X_2, AX_2X_3, ... DEY_1, DY_1Y_2, DY_2Y_3, ...PQR, also being equal, the triangles are each of the same area.

$\triangle ABC$ contains p triangles each equal to $\triangle PQR$ in area and $\triangle DEF$ contains q triangles each of the same area.

$\therefore$ Considering $\triangle PQR$ as the unit of measurement,

$$\triangle ABC : \triangle DEF = p : q = BC : EF.$$

Notes. (1) A parallelogram has twice the area of the triangle on the same base and having the same altitude. Hence this theorem holds good if *parallelograms* be substituted for *triangles* in the theorem.

(2) If k be the common altitude of the triangles, we may write

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC.k}{\frac{1}{2}EF.k} = \frac{BC}{EF},$$ which proves the result.

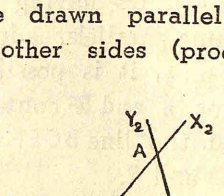
(3) This theorem may be slightly altered to give another theorem, *viz.*, the areas of triangles on equal bases are as their altitudes. For, if ABC and DEF be two triangles whose bases BC and EF are equal, and if their altitudes are k and l, then

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC.k}{\frac{1}{2}EF.l} = \frac{k}{l},$$ which is the result.

(Prove this theorem directly with the aid of a diagram.)

Theorem 15.

A straight line drawn parallel to a side of any triangle cuts the other sides (produced if necessary) proportionally.



Let ABC be a triangle and XY a straight line parallel to BC meeting AB in X and AC in Y. It is required to show that AX : XB = AY : YC. Join BY and XC.

The triangles AXY, BXY have a common vertex at Y. AX, XB are their bases and their altitudes are equal.

$$\therefore \triangle AXY : \triangle BYX = AX : XB.$$

Similarly $\triangle AXY : \triangle CYX = AY : YC$.

But $\triangle BYX = \triangle CYX$, since XY is parallel to BC.

$\therefore AX : XB = AY : YC$.

Notes. (1) The proof is substantially the same whether the line parallel to BC cuts the sides internally as at X and Y, or externally as at X_1 and Y_1 or X_2 and Y_2.

(2) From the proportion $AX : XB = AY : YC$, we get by componendo, $AB : XB = AC : YC$ and by division of these equal ratios, $AB : AX = AC : AY$.

$\therefore$ The segments AX, XB and AB of one line are respectively proportional to the corresponding segments AY, YC and AC of the other line. These corresponding segments are also said to be *homologous*.

(3) The *converse* is also true, *viz.*, if X, Y are points on the sides AB, AC of a triangle ABC, such that $AX : XB = AY : YC$, then XY is parallel to BC.

For, as before, $\triangle AXY : \triangle BYX = AX : XB$,

$\triangle AXY : \triangle CYX = AY : YC$.

If $AX : XB = AY : YC$, then $\triangle BYX = \triangle CYX$.

$\therefore$ XY is parallel to BC.

(4) The theorem may also be proved as follows: If $AX : XB = l : m$, it must be possible to find a length which is contained l times in AX and m times in XB. Dividing AB into $l+m$* parts of such length and drawing parallels to BC through the points of division, it will be found that AC will also be divided into $l+m$ equal parts, AY containing l and YC containing m of these parts. Hence $AX : XB = l : m = AY : YC$.

The *converse* is proved as follows:

If XY is not parallel to BC, let XZ be drawn parallel to BC so as to cut AC in Z.

Then $AZ : ZC = AX : XB = AY : YC$.

$\therefore AZ : ZC = AY : YC$.

$\therefore$ Z coincides with Y, or XY is parallel to BC.

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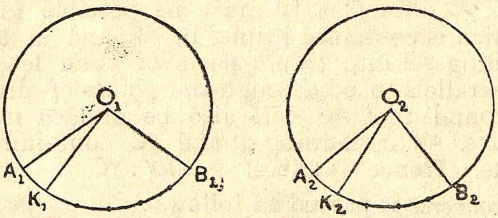
* X *is here assumed to lie within* AB.

(5) This theorem is useful to obtain the fourth proportional to three numbers by a geometrical construction. For, in the diagram, since AX : XB = AY : YC, YC is the fourth proportional to AX, XB, AY. The following construction gives YC. Draw a line AXB so that AX, XB are equal to the first two proportionals. Draw any other convenient line through A and take a segment AY in it equal to the third proportional. Join XY and draw BC parallel to it so as to meet AY in C. Then YC is the required fourth proportional. Similarly AC is the fourth proportional to AX, AB, AY.

(6) This gives a construction for dividing a given line in a given ratio. Let AB be the given line. Draw any convenient line through A and along it step off AY, YC equal to l, m units of any convenient magnitude. Join BC and draw XY parallel to it so as to cut AB in X. Then AX : XB = l : m. Note that Y lies between A and C for internal division and outside AC for external division.

Theorem 16.

In equal circles, angles, whether at the centres or at the circumferences, have the same ratio as the arcs on which they stand.



Let A_1B_1 and A_2B_2 be two arcs of equal circles whose centres are O_1 and O_2 and let arc A_1B_1 : arc $A_2B_2 = l : m$. Then it is possible to determine a common measure which is contained l times in the arc A_1B_1 and m times in the arc A_2B_2. Let A_1K_1 and A_2K_2 be two arcs in the circles, each equal to that common measure.

The arc A_1B_1 can be divided into l arcs each equal to the arc A_1K_1 and since equal arcs subtend equal angles at the centres, $\angle A_1O_1B_1$ is l times $\angle A_1O_1K_1$. Similarly $\angle A_2O_2B_2 = m.\ \angle A_2O_2K_2$. But $\angle A_1O_1K_1 = \angle A_2O_2K_2$, since they are angles subtended by equal arcs of equal circles.

$$\therefore \angle A_1O_1B_1 : \angle A_2O_2B_2 = l : m = \text{arc } A_1B_1 : \text{arc } A_2B_2.$$

The angle at the circumference subtended by any arc being half the angle subtended at the centre, it follows that the angles at the circumferences are also in the same ratio.

Notes. (1) The theorem applies also to the arcs of *the same circle.*

(2) The sectors on equal arcs of equal circles being equal in area, the areas of sectors of equal circles are in the ratio of the arcs on which they stand.

(3) Chords are *not* proportional to the arcs they cut off or to the angles they subtend at the centre or at the circumference. In fact, by a theorem in Trigonometry, *the chords of a circle are proportionnl to the sines of the angles they subtend at the circumference.*

(4) If the angle subtended at the centre of a circle by an arc of the circumference equal in length to the radius is defined as a *radian*, then by this theorem, the radian measure θ, of the angle subtended at the centre of a circle of radius r by an arc of length s, is given by the formula $\theta : 1 = s : r$ or $s = r\,\theta$.

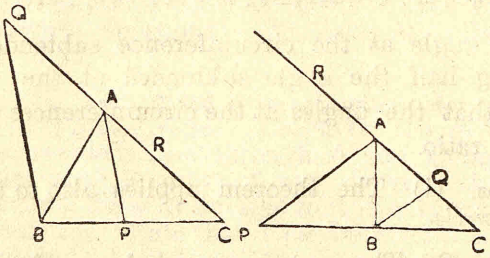
The measure of the radian itself is again given by this formula. For, if x degrees be the magnitude of a radian, $x : 180 = r : \pi r$, since half the circumference (πr) subtends 180° at the centre.

$$\therefore x = 180/\pi = 180/3{\cdot}1416 \text{ nearly.}$$

On simplification, it is found that a radian contains 57·2958 degrees or 57° 17′ 45″ approximately.

Theorem 17.

If the vertical angle of a triangle is bisected internally or externally by a straight line which also cuts the base, then the base is divided internally or externally in the ratio of the sides of the triangle.



Let AP be the bisector of the angle A of the triangle ABC, meeting BC in P. Draw BQ parallel to AP to cut AC, produced if necessary, at Q. Let R be a point within AC if Q lies in CA produced, or a point in CA produced if Q lies within CA.

Since BQ is parallel to AP,

$$\angle BAP = \angle ABQ \quad \text{and} \quad \angle RAP = \angle AQB.$$

But $\angle BAP = \angle RAP$. $\therefore \angle ABQ = \angle AQB$ and $AB = AQ$.

Also, since BQ is parallel to AP,

$$QA : AC = BP : PC.$$

$$\therefore BP : PC = AB : AC, \qquad \text{since } QA = AB.$$

Notes. (1) The proof is the same for both the cases, provided the letters are properly marked. The theorem excludes the case of the external bisector when $AB = AC$.

(2) The *converse* is also true, *viz.*, if P be a point in the base BC of a triangle ABC, such that $BP : PC = AB : AC$, then AP bisects the vertical angle internally or externally.

For, with the same construction,

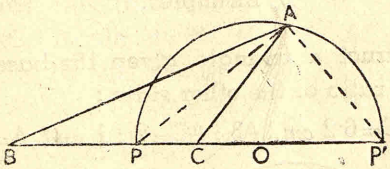
$BP : PC = QA : AC$ and $\therefore QA = AB$ and $\angle ABQ = \angle AQB$.

But since $\angle ABQ = \angle BAP$ and $\angle AQB = \angle RAP$,

$\angle BAP = \angle RAP$, and AP bisects the angle A.

(3) If P, P′ are the points in which the internal and external bisectors of $\angle A$ meet the base, $\angle PAP' = 90°$. When the base and the ratio of the other sides are given, the positions of P and P′ on the line BC may be determined by dividing the base internally and externally in the ratio of the sides. The position of A lies on the circle on PP′ as diameter since $\angle PAP' = 90°$.

(4) *Conversely* if a straight line BC is divided internally at P and externally at P′ in the ratio $l : m$ and if any point A is taken on the circle on PP′ as diameter, then $AB : AC = l : m$. (*m*)



Let O be the centre of the circle.

$BP : PC = BP' : CP'$. $\therefore BP' : BP = CP' : PC$.

By componendo and dividendo,

$$\frac{BP' + BP}{BP' - BP} = \frac{CP' + PC}{CP' - PC}, \text{ or } \frac{2\,OB}{2\,OP} = \frac{2\,OP}{2\,OC}.$$

$$\therefore OC.OB = OP^2 = OA^2.$$

$\therefore$ OA is a tangent to the circle ABC.

$$\therefore \angle OAC = \angle OBA \text{ or,}$$

$$\angle OAP - \angle PAC = \angle OPA - \angle BAP = \angle OAP - \angle BAP.$$

$\therefore \angle BAP = \angle PAC$ and AP bisects $\angle BAC$ internally.

$$\therefore AB : AC = BP : PC = l : m.$$

(5) The circle on PP′ as diameter, which is the *locus* of the point A whose distances from the fixed points B and C are in a constant ratio (BP : PC = BP′ : CP′) is called a **circle of Apollonius.**

(6) If $BC=a$, $CA=b$, $AB=c$, and AP is the internal bisector, $BP:PC=c:b$. ∴ By componendo, $BP+PC:PC=b+c:b$ or $BC:PC=b+c:b$.

$\therefore PC\,(b+c)=ab$, or $PC=ab/(b+c)$. $\therefore BP=ac/(b+c)$.

Similarly if the external bisector meets the base in P′

$BP'=ac/(b-c)$, $CP'=ab/(b-c)$, if $AC>AB$,

or $BP'=ac/(c-b)$, $CP'=ab/(c-b)$, if $AB>AC$.

If $AC>AB$,

$PP'=CP'-CP=ab/(b-c)-ab/(b+c)=2abc/(b^2-c^2)$

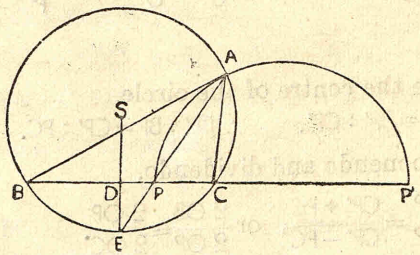
and if $AB>AC$,

$PP'=BP'-BP=ac/(c-b)-ac/(c+b)=2\,abc/(c^2-b^2)$.

Examples.

(1) **Construct a triangle given the base, the vertical angle and the ratio of the other sides;**

for example, $BC=6{\cdot}2$ *cm.*, $AB:AC=2:1$ *and* $A=51°$.

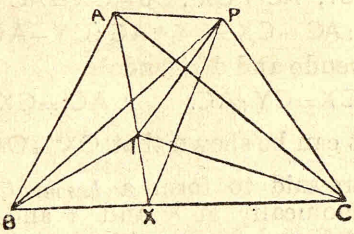


On the base BC describe the segment of a circle containing the given vertical angle. Divide BC internally at P and externally at P′ in the ratio of the sides. Describe the circle of Apollonius on PP′ as diameter, so as to cut the former segment of the circle at A. Then ABC is the required triangle.

Otherwise, draw BC and the circum-circle as before. Divide BC internally at P in the ratio of the sides. Join E the mid-point of the arc BC to P and produce the line to meet the circle at A. ABC is the required triangle.

(2) If P be any point in the plane of a triangle ABC, and I the in-centre, show that

$$a.PA^2 + b.PB^2 + c.PC^2 = (a+b+c)\ PI^2 + a.AI^2 + b.BI^2 + c.CI^2$$



If AI meets the base in X, $BX : XC = AB : AC = c : b$, or $b.BX = c.XC$. Again $AI : IX = AB : BX = AC : XC = AB + AC : BX + XC = b + c : a$, by the *theorem of equal ratios.*

$$\therefore a.AI = (b+c).IX.$$

$\because b.BX = c.XC$, we get from $\triangle BPC$,

$$b.BP^2 + c.CP^2 = (b+c)\ PX^2 + b.BX^2 + c.XC^2 \qquad \text{(Th. 10)}$$

$$\therefore a.AP^2 + b.BP^2 + c.CP^2 = a.AP^2 + (b+c)\ PX^2 + b.BX^2 + c.XC^2.$$

$\because a.AI = (b+c)\ IX$, from $\triangle APX$,

$$a.AP^2 + (b+c)\ PX^2 = (a+b+c)\ PI^2 + a.AI^2 + (b+c)\ IX^2.$$

and again from $\triangle BIC$ in which $b.BX = c.XC$,

$$b.BI^2 + c.CI^2 = (b+c)\ IX^2 + b.BX^2 + c.XC^2.$$

$$\therefore a.AP^2 + b.BP^2 + c.CP^2 = (a+b+c)\ PI^2 + a.AI^2 + b.BI^2 + c.CI^2$$

which has a minimum value when P coincides with I.

(3) If a straight line AB is divided internally at X and externally at Y in the same ratio, show that

$$1/AX + 1/AY = 2/AB = 1/XB - 1/BY,$$

If C and O are the mid-points of AB and XY, prove that

$$CA^2 = CX.CY \quad \text{and} \quad OX^2 = OA.OB.$$

A C X B O Y

$$\frac{XB}{AX}=\frac{BY}{AY} \text{ or } \frac{AB}{AX}-1=1-\frac{AB}{AY}. \quad \therefore \frac{AB}{AX}+\frac{AB}{AY}=2.$$

$$\therefore 1/AX+1/AY=2/AB.$$

$$\text{Again } \frac{AX}{XB}=\frac{AY}{BY} \text{ or } \frac{AB}{XB}-1=1+\frac{AB}{BY}. \quad \therefore \frac{AB}{XB}-\frac{AB}{BY}=2.$$

$$\therefore 1/XB-1/BY=2/AB.$$

$\because AX : XB = AY : BY$, $AC+CX : CB-CX = AC+CY : CY-CB$.

$\therefore AC+CX : AC-CX = CY+AC : CY-AC$. ($\because CB = AC$)

By componendo and dividendo

$$AC : CX = CY : AC. \qquad \therefore AC^2 = CX.CY.$$

Similarly it can be shown that $OX^2 = OA.OB$.

A, X, B, Y are said to form a *harmonic range* and AB is divided harmonically at X and Y and XY is divided harmonically at A and B.

Exercise 5

(1) A straight line AB 4·8″ long is divided at X internally in the ratio 5 : 3. Calculate the lengths of the segments AX and XB and verify the results by a construction.

(2) A straight line AB 2·7″ long is divided at Y externally in the ratio 5 : 2. Calculate the lengths of AY and BY and verify the results by drawing a diagram to scale.

(3) A straight line AB 5·2 cm. long is divided internally at X and externally at Y in the ratio 3 : 7. Calculate the lengths of AX, XB, YA, YB and verify the formula

$$1/AX - 1/YA = 2/AB.$$

(4) XY is drawn parallel to the side BC of a triangle ABC, cutting AB and AC at X and Y respectively.

AB = 3·6 cm., BC = 6·0 cm., AC = 4·8 cm.

(i) If AX = 2·4 cm., calculate YC, XY.
(ii) If XB = 1·6 cm., calculate AY.
(iii) If YC = 1·8 cm., calculate AX and the area of $\triangle AXY$.
(iv) If AY = 1·2 cm., calculate the ratio $\triangle AXY : \triangle ABC$.

(5) ABC is a triangle in which AB = 3·2 in., A = 90°, AC = 2·8 in. AB is divided externally at X in the ratio 5 : 1 and XY is drawn parallel to BC so as to cut AC produced at Y. Calculate AX, AY and the ratio $\triangle$ AXY : $\triangle$ ABC.

(6) Show that three parallel straight lines cut any two transversals proportionally.

(7) The straight line joining the mid-points of the oblique sides of a trapezium is parallel to the other sides.

(8) In a triangle ABC, AX is drawn perpendicular to the bisector of the angle B. If the line through X parallel to BC is drawn, show that it will bisect AB and AC.

(9) D is the mid-point of the base BC of a triangle ABC and E divides CA in the ratio 2 : 3. If BF is drawn parallel to DE and meets AC in F, show that AC = 5 AF. If ED, AB meet in G, show that AG = 3 AB.

(10) If two circles touch each other at P, externally or internally, show that any two lines through P will be cut proportionally by the circumferences.

(11) Show that the internal bisectors of the angles of a triangle meet in a point. Prove that the internal bisector of any angle and the external bisectors of the remaining angles of a triangle also meet in a point.

(12) The bisectors of the angles A and C of a quadrilateral ABCD meet on the diagonal BD. Show that the bisectors of the angles B and D meet on AC. (*m*)

(13) If the internal bisector of the angle A of a triangle ABC meets the opposite side in P show that

$$b.\,BP^2 + c.\,CP^2 + (b+c).\,AP^2 = bc\,(b+c).$$

Hence deduce the length of the bisector of A in the form

$$AP = \sqrt{\left[bc\left\{ 1 - \frac{a^2}{(b+c)^2} \right\} \right]} = \frac{2\,bc}{b+c} \cos \frac{A}{2}. \qquad (m)$$

(14) ABC is any triangle and AD, AP, AP', AX are respectively the median, the internal and the external bisectors of A, and the altitude through A, meeting BC in D, P, P', X. BK is the perpendicular from B to AP. Show that (i) B, A, K, X are concyclic,

(ii) $DP.DX = DK^2 = \frac{1}{4}\,(b \sim c)^2$ (iii) $DP.DP' = BD^2 = DC^2$.

(15) Construct the triangle ABC having

(i) BC = 3·2″, AB : AC = 7 : 4, A = 75°.

(ii) BC = 6·4 cm., AB : AC = 2 : 5, the median AD = 4·1 cm.

(iii) BC = 2·3″, AB : AC = 8 : 3, the altitude AX = 1·8″.

(iv) BC = 6·7 cm., AB : AC = 3 : 7, AP the internal bisector of A = 4·9 cm.

(v) BC = 3·6″, AB : AC = 9 : 4, the median BE = 3·2″.

(16) Describe two circles of radii 1·4″ and 1.8″ with their centres 2·5″ apart. Draw a straight line through one of their points of intersection so that the chords intercepted may be in the ratio 3 : 5.

2. Similar Triangles.

Def. Two rectilineal figures having the same number of sides are said to be **similar** if they are equiangular and if the sides about every pair of equal angles are proportional.

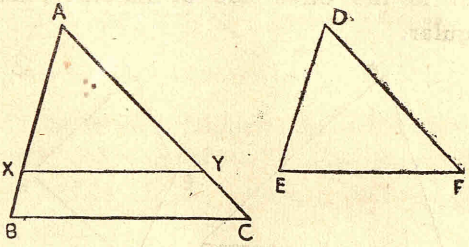
Notes. (1) The similarity lies in the fact that the figures have the same shape. For example, there is an obvious similarity in the shapes of two maps of India, one a wall-map and the other, the map in a book. The sizes are not necessarily the same but the figures have the same shape. When the shapes and the sizes are the same, the figures are equal in all respects and one can be superposed on the other so that they coincide.

(2) If the figures are equiangular without the correspoding sides being proportional, they are not similar. For example, a square and a rectangle are equiangular but they are not of the same shape. Again if the corresponding sides are proportional but the figures not equiangular, they are not similar; *e.g.*, a square and a rhombus have their sides proportional but they are not similar. But all squares are similar.

Hereafter it will be shown that in the case of two *triangles*, either of these conditions is sufficient for similarity.

Theorem 18.

If two triangles are equiangular, their corresponding sides are proportional.



Let ABC and DEF be two triangles in which

$$\angle A = \angle D, \quad \angle B = \angle E \quad \text{and} \quad \angle C = \angle F.$$

In AB take the point X so that AX = DE, and in AC take the point Y so that AY = DF. Then the triangles AXY, DEF are equal in all respects.

$$\therefore \angle AXY = \angle DEF = \angle ABC.$$

$\therefore$ XY is parallel to BC and cuts AB and AC proportionally.

$$\therefore AX : AB = AY : AC$$

$$\text{or} \quad DE : AB = DF : AC.$$

Similarly it can be shown that each of these ratios is equal to EF : BC.

Thus AB, BC, CA are proportional to DE, EF, FD respectively.

(*The triangles* ABC *and* DEF *are similar.*)

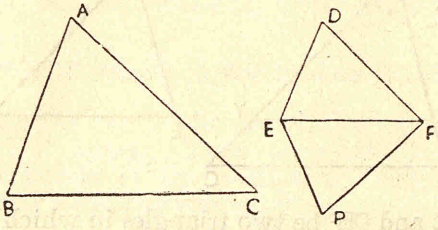
Notes. (1) The corresponding sides of two similar triangles are those which are opposite to equal angles.

(2) Similar triangles can be placed as in the diagram so that their corresponding sides are parallel. When they can be so placed without turning over the face of the triangle, they are said to be *directly imilar*, otherwise they are *inversely similar.*

(△^{s} DEF and EPF on the next page are inversely similar.)

Theorem 19.

If two triangles have the three sides of the one proportional to the three sides of the other, the triangles are equiangular.



Let ABC, DEF be two triangles in which AB : DE = BC : EF = AC : DF. On the side of EF opposite to that of D, construct △ EPF equiangular with △ ABC, by taking

$$\angle PEF = \angle ABC, \quad \angle PFE = \angle ACB.$$

Then $\angle EPF = \angle BAC$.

$\therefore$ By construction, $AB : PE = BC : EF = AC : PF$.

By hypothesis, $AB : DE = BC : EF = AC : DF$.

$\therefore PE = DE$ and $PF = DF$. $\therefore$ △ PEF ≡ △ DEF.

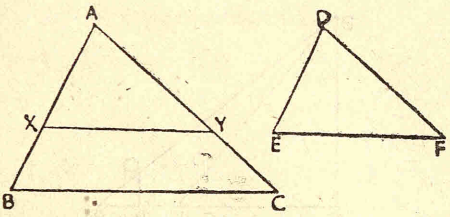
$\therefore$ △ DEF is equiangular with △ PEF which itself is equiangular with △ ABC.

$\therefore$ △ DEF and △ ABC are equiangular and similar, the angles A, B, C being respectively equal to the angles D, E, F.

Note. This is the *converse* of the previous theorem.

Theorem 20

If two triangles have an angle of the one equal to an angle of the other, and the sides about those equal angles proportional, the triangles are similar.



Let ABC, DEF be two triangles in which $\angle A = \angle D$ and $AB : DE = AC : DF$. Along AB, AC take segments AX, AY respectively equal to DE, DF. Then $\triangle$ AXY is equal in all respects to $\triangle$ DEF.

By hypothesis and construction, $AB : AX = AC : AY$.

$\therefore$ XY is parallel to BC.

$\therefore \angle B = \angle AXY = \angle E$ and $\angle C = \angle AYX = \angle F$.

$\therefore$ The triangles ABC, DEF are equiangular and similar.

Notes. (1) The theorems 18, 19 and 20 state the alternative conditions under which two triangles are similar. A special case of right-angled triangles will be considered later, (Theorem 21, Note 4). The student is advised to recollect the theorems on the congruence of triangles, corresponding to these.

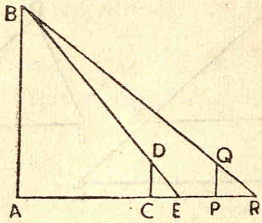
(2) Two triangles are similar if their corresponding sides are parallel or perpendicular, for then they become equiangular. For example, the middle point triangle formed by joining the mid-points of the sides of any triangle is similar to the orginal triangle.

(3) '*Parallelograms*' may be substituted for '*triangles*' in the theorem.

Examples.

(1) *A man whose height is 5 ft. stands at a distance of 16 ft. from a lamp-post 25 ft. high. How much farther away from the lamp-post must he move, so that his shadow may become* $1\frac{1}{2}$ *times as long as at present?*

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Let AB represent the lamp-post and CD the man in his first position, CE being his shadow. AB=25, CD=5, AC=16. Let CE=x.

The triangles ABC, CDE are equiangular and similar.

$\therefore$ AB : CD = AE : CE, or $25 : 5 = (16+x) : x$.

$\therefore 16+x=5x$ and $x=4$.

The present length of the shadow is 4 ft.

Let the man now move into the position PQ, where the shadow PR is 6 ft. in length. Then $\triangle$ ABR and $\triangle$ PQR are equiangular and similar.

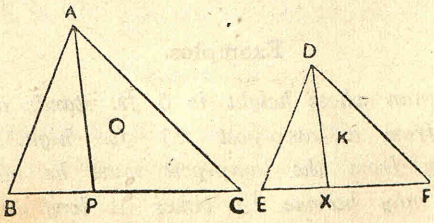
$\therefore$ AB : PQ = AR : PR or $25 : 5 = AR : 6$. $\therefore$ AR = 30.

$\because$ PR = 6, AP = AR − PR = 24.

But AC = 16. $\therefore$ CP = 8.

The man must move 8 ft. away from the lamp-post.

(2) In two similar triangles, the ratios of the corresponding medians, altitudes, bisectors of the angles, circum-radii and in-radii, are each equal to the ratio of the corresponding sides.



Let ABC, DEF be two similar triangles.

(i) Let AP, DX be the corresponding medians of the triangles.

In $\triangle^s$ ABP and DEX, $\angle B = \angle E$ and the sides about these angles are proportional since $BP : EX = \frac{1}{2}BC : \frac{1}{2}EF = AB : DE$.

$\therefore$ The triangles are similar and $AP : DX = AB : DE$.

(ii) If AP, DX denote the corresponding altitudes, the triangles ABP, DEX are equiangular and similar.

$$\therefore AP : DX = AB : DE.$$

(iii) If AP, DX are respectively the bisectors (internal or external) of the angles A and D, then $\angle ABP = \angle DEX$, $\angle BAP = \angle EDX$ (these being halves of equal angles) and $\therefore \angle APB = DXE$ and the triangles are equiangular.

$$\therefore AP : DX = AB : DE.$$

(iv) Let O, K be the circum-centres of $\triangle^s$ ABC, DEF.

Then $\angle BOC = \angle EKF$, ($\because$ they are double $\angle A$ and $\angle D$ respectively) and $BO : OC = 1 : 1 = EK : KF$.

$\therefore$ The triangles BOC, EKF are similar and

$$BO : EK = BC : EF.$$

$\therefore$ The circum-radii are proportional to the sides.

(v) If O, K are in-centres of $\triangle^s$ ABC, DEF, the triangles BOC, EKF are equiangular since

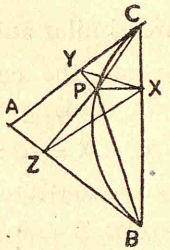
$$\angle OBC = \tfrac{1}{2}B = \tfrac{1}{2}E = \angle KEF, \quad \angle OCB = \tfrac{1}{2}C = \tfrac{1}{2}F = \angle KFE$$

$\therefore$ The altitudes of those triangles are proportional to their sides BC, EF.

$\therefore$ The in-radii are proportional to the sides.

Note. Similarly it follows that **the ratio of the distances between any two corresponding pairs of points of the similar triangles is equal to the ratio of the corresponding sides.**

(3) The locus of a point, such that the rectangle contained by its distances from the equal sides of an isosceles triangle is equal to the square on its distance from the third side, is the circle touching the equal sides at the extremities of the base.



Let PX, PY, PZ be perpendicular to the sides BC, CA, AB of a triangle ABC in which AB=AC. Join BP, PC, XY, XZ.

(i) (P, X, C, Y) and (P, X, B, Z) are two sets of concyclic points.

$\therefore \angle YPX = 180° - \angle ACB = 180° - \angle ABC = \angle ZPX.$

If $PX^2 = PY.PZ$, $PY : PX = PX : PZ$.

$\therefore \triangle^s PXY, PXZ$ are similar and hence equiangular.

$\therefore \angle YCP = \angle YXP$ standing on the same arc PY
$= \angle XZP$
$= \angle XBP$ standing on the arc PX.

$\therefore$ AC is the tangent at C to the circle CPB or P lies on the circle which touches AC at C and which also similarly touches AB at B.

(ii) If P be any point on the circle which touches AB at B and AC at C, then

$\angle YXP = \angle YCP$ standing on the arc PY
$= \angle XBP$ in the alternate segment
$= \angle XZP$ standing on the arc PX.

Similarly $\angle XYP = \angle ZXP$ and $\therefore \angle XPY = \angle XPZ$.

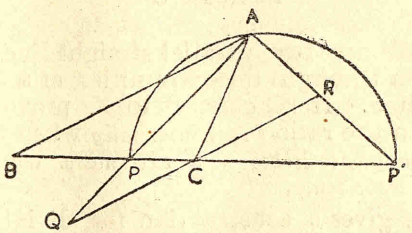
The triangles PXY, PXZ are equiangular and similar.

$\therefore PY : PX = PX : PZ$ and $PX^2 = PY.PZ$.

(iii) Every point P such that $PX^2 = PY.PZ$ lies on the circle touching AB and AC at B and C respectively; and every point P on that circle satisfies the condition $PX^2 = PY.PZ$. Therefore the *locus* of P is the circle touching AB and AC at B and C respectively.

(4) **If a straight line BC is divided internally at P and externally at P′ in the ratio $l : m$ and if any point A is taken on the circle on PP′ as diameter, then $AB : AC = l : m$.**

(See Note 4 to Th. 17)



Draw QR ∥ to AB through C, to cut AP at Q and AP′ at R

$\triangle^s$ ABP, PQC are equiangular. $\therefore BP : PC = AB : QC.$

$\triangle^s$ ABP′, RCP′ are equiangular. $\therefore BP' : P'C = AB : CR.$

$\because BP : PC = BP' : P'C, \quad QC = CR.$

$\because \angle QAR = 90°$ and $QC = CR$, each is equal to AC.

$\therefore AB : AC = BP : PC = l : m.$

(3) **The corresponding segments of two parallel straight lines cut off by three or more concurrent transversals are proportional.**

(Draw a suitable figure.)

If a number of lines concurrent at O meet one transversal in the the points A, B, C, D,.....and the other transversal parallel to the former in the points A′, B′, C′, D′..... then (OA′ B′, OAB), (OB′ C′, OBC), (OC′ D′, OCD),.... are pairs of equiangular and similar triangles.

$$\therefore \frac{A'B'}{AB} = \frac{OB'}{OB} = \frac{B'C'}{BC} = \frac{OC'}{OC} = \frac{C'D'}{CD} = \ldots\ldots$$

Conversely, (i) *If the intercepts made by three or more concurrent lines on two given lines are proportional, the lines are parallel and* (ii) *if the segments* A′B′, B′C′, C′D′,.....*on a straight line are respectively proportional to the corresponding segments* AB, BC, CD,.....*on a parallel straight line, then* AA′, BB′, CC′, DD′,..... *are concurrent.*

(These results are proved by *Reductio ad Absurdum* and are left as exercise to the student.)

Exercise 6

(1) AP, BQ are two parallel straight lines of lengths l and m drawn through the extremities of a segment AB of a straight line. If PQ cuts AB in X, prove that AB is divided at X in the ratio $l : m$, *internally* when the parallels are drawn in *opposite senses* and *externally* when they are drawn in the *same sense.*

(This gives a construction for dividing a given line in a given ratio)

(2) P and Q are any two points on the arms of an angle AOB. If PM and QN are drawn perpendicular to the arms not containing P and Q respectively, prove that

$$OP : OQ = PM : QN = OM : ON$$

This result is the basis of Trigonometry, for it shows that a trigonometrical ratio such as PM : OP is constant for a given angle and does not depend upon the position of P.

(3) If AB, CD are two chords of a circle intersecting at O, prove that the pairs of triangles (AOC, BOD), (AOD, COB) are similar. Hence deduce the result of Theorem 11, *viz.*, $AO \cdot OB = CO \cdot OD$.

(4) If OAB is a secant and OT a tangent drawn from O to any circle, show that OTA and OTB are similar triangles and hence deduce that $OT^2 = OA \cdot OB$. (Th. 12)

(5) If AB and CD meet at O when produced and if $OA \cdot OB = OC \cdot OD$, prove that A, C, B, D are concyclic. If T be a point such that $OT^2 = OA \cdot OB$, show that OT is a tangent to the circle ABT.

(6) The chords of two circles which subtend equal angles either at the centres or at the circumferences are in the ratio of the diameters of the circles.

(7) Show that the chords of a circle are proportional to the sines of the angles they subtend at the circumference. Hence show that, in any triangle,

$$a : \sin A = b : \sin B = c : \sin C.$$

(8) The diagonals of a quadrilateral ABCD intersect in O. The circles OBC, ODA intersect again in P. Show that the triangles PBC and PDA are similar.

(9) PM and QN are the perpendiculars from two given points P and Q to a given straight line AB. PN and QM intersect in R and RS is drawn perpendicular to AB. Show that PS and QS are equally inclined to AB.

(10) AP the bisector of the angle A of a triangle ABC is produced to meet the circum-circle at Q. Show that ABP, AQC are equiangular triangles and deduce that $AB \cdot AC = AP \cdot AQ$. Also prove that $BQ^2 = CQ^2 = QP \cdot QA$. (Th. 31)

(11) X is a point on the base BC of an isosceles triangle ABC, such that $AB^2 = AC^2 = BX^2 = BC \cdot XC$. Prove that ABC and AXC are equiangular triangles and $A = 3B = 3C$.

(12) If AX, BY, CZ are drawn perpendicular to the sides BC, CA, AB of a triangle ABC, show that the triangles AZY, BXZ, CYX are equiangular with ABC.

(13) Show that the perimeters of the regular polygons of the same number of sides inscribed in two circles are in the ratio of their radii.

(14) In a triangle ABC, I is the centre of the inscribed circle whose radius is r and I_1 the centre of the escribed circle opposite to A whose radius is r_1 Prove that $AI : AI_1 = r : r_1$

(15) A man whose height is 6 ft. stands 32 ft. from a lamp-post and observes that his shadow cast by the light is 8 ft. in length. How high is the light above the ground and how long would be the shadow of a man 5 ft. in height, standing 20 ft. from the lamp-post ?

(16) Prove that the distance of the centroid of a triangle from any side is one-third of the distance of the opposite vertex from that side.

(17) A straight line parallel to the tangent at A to the circum-circle of a triangle ABC meets the sides AB and AC in X and Y respectively. Prove that $AB.AX = AC.AY$.

(18) Two circles of radii r and r' have external contact at A and a common tangent touches them at P and Q. Show that PAQ is a right angle and that $PQ^2 = 4rr'$,

(19) AB is a diameter of a circle and C is any point on the circumference. From any point P in AB, PE is drawn perpendicular to AB, meeting the circumference in F and CA, CB in D, E respectively. Prove that $PF^2 = PD.PE$.

(20) AB is a diameter of a circle ACB and the tangents at A and B meet the tangent at any point C on the circumference at D and E. Prove that BD bisects the perpendicular from C to AB.

(21) The diagonals of a quadrilateral ABCD intersect in O and the circles OAB, OCD meet again at E. Show that EAB, ECD are similar triangles.

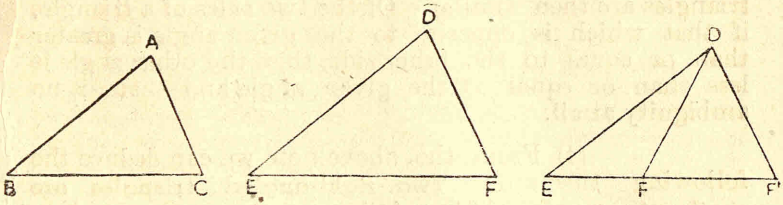
(22) The tangent at A to the circum-circle of a triangle ABC cuts BC produced in D. Show that BC is divided externally at D in the ratio $AB^2 : AC^2$. (m)

(23) AB, BC are two sides of a regular pentagon inscribed in a circle of radius r. P is the mid-point of the arc BC. Show that $AP.BP = r^2$.

(24) If A, B, C, D are four consecutive vertices of a regular heptagon, prove that $AD^2 = AC\ (AB + AC)$.

Theorem 21.

If two sides of one triangle are proportional to two sides of another triangle and if the angles opposite to one pair of corresponding sides are equal, then the angles opposite to the other pair are either equal or supplementary.



Let ABC, DEF be two triangles in which AB : DE = AC : DF and $\angle B = \angle E$. It is required to prove that $\angle C$ and $\angle F$ are either equal or supplementary.

Case (i). If $\angle A = \angle D$, then $\angle C = \angle F$.

Case (ii). If $\angle A$ is not equal to $\angle D$ as in the extreme figures, draw DF' making $\angle EDF'$ equal to $\angle BAC$ and let DF' meet EF in F'.

Then by construction, ABC, DEF' are equiangular triangles, and therefore AB : DE = AC : DF'.

By hypothesis, AB : DE = AC : DF.

$\therefore$ DF' = DF and $\angle DFF' = \angle DF'F$.

$\therefore \angle EFD = 180° - \angle DFF' = 180° - \angle DF'E = 180° - \angle ACB$.

$\therefore$ C and F are supplementary angles.

Notes. (1) This corresponds to the following theorem under congruence of triangles: *If two triangles have two sides of the one equal to two sides of the other and if the angles opposite to one pair of corresponding sides are equal, then the angles opposite to the other pair are either equal or supplementary.*

(2) In case the second corresponding pair of angles C and F are equal, the triangles are equiangular and similar.

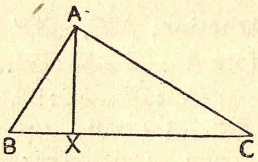
(3) There is in general a doubt or *ambiguity* about the relationship of these angles, except when something more is known about the triangles. For example, if the given equal angles are greater than or equal to a right angle, the other pair can only be equal and both acute and not supplementary at all. The

triangles are then similar. Of the two sides of a triangle, if that which is opposite to the given angle is greater than or equal to the other side, then the other angle is less than or equal to the given angle and there is no ambiguity at all.

(4) From the above note we can deduce the following theorem: **Two right-angled triangles are similar if *any* two sides of the one are proportional to the corresponding sides of the other.**

Theorem 22.

The perpendicular drawn from the right angle to the hypotenuse of a right-angled triangle divides the triangle into two parts which are similar to each other and to the original triangle.



Let ABC be a triangle right angled at A and let AX be drawn perpendicular to BC and meeting it in X.

$\angle BAX = 90° - \angle XAC = \angle ACX$, and similarly $\angle XAC = \angle ABX$.

$\therefore$ In the triangles ABC, ABX, ACX,

$$\angle ABC = \angle ABX = \angle XAC$$
$$\angle ACB = \angle BAX = \angle ACX$$
$$\angle BAC = \angle AXB = \angle AXC$$

The three triangles are equiangular and similar when taken in pairs.

Notes. (1) From the similar triangles ABC, ABX we have

$$AB : BX = BC : AB = CA : AX.$$

From $\triangle^s$ ABC, ACX, $AB : AX = BC : AC = CA : XC$.

From $\triangle^s$ ABX, ACX, $AB : AC = BX : AX = AX : XC$.

(2) From these we find that (i) $AB^2 = BX.BC$, (ii) $AC^2 = BC.XC$, (iii) $AX^2 = BX.XC$, or that AB is the mean proportional between BX and BC, AC is the mean proportional to BC and XC and AX is the mean proportional to BX and XC. This gives a geometrical construction for finding the mean proportional between two given magnitudes. For example if the line BXC is drawn so that BX, XC represent the given magnitudes and if the perpendicular to BC through X meets the semi-circle on BC at A, then AX represents the mean proportional.

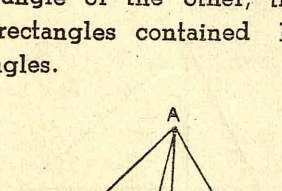
(See construction 1.)

(3) From the above equations,

$$AB^2 : AC^2 = BX : XC.$$

Theorem 23.

If two triangles have one angle of the one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about those equal angles.



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If ABC, DEF are two triangles in which $\angle B = \angle E$, it is required to prove that $\triangle ABC : \triangle DEF = AB.BC : DE.EF$.

Let AX and DY be the altitudes of the triangles.

The triangles ABX, DEY are equiangular and similar.

$$\therefore AX : DY = AB : DE.$$

$$\text{Also } \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} BC.AX}{\frac{1}{2} EF.DY} = \frac{BC}{EF} \cdot \frac{AX}{DY} = \frac{BC}{EF} \cdot \frac{AB}{DE} = \frac{AB.BC}{DE.EF}.$$

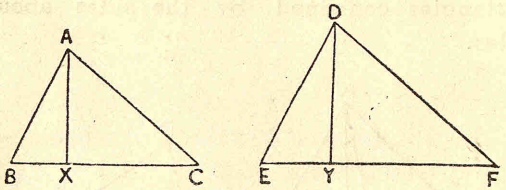
Notes. (1) The word *triangles* of the theorem may be also replaced by the word *parallelograms.*

(2) Using the trigonometrical expression $\frac{1}{2}\, ca \sin B$ for the area of a triangle, we see that the areas of the two triangles ABC, A′B′C′ are as $ca : c'a'$ if $B = B'$.

(3) From the above note, since $\sin B = \sin (180° - B)$, it follows that *the areas of two triangles are proportional to the rectangles contained by their corresponding pairs of sides, if the angles between the pairs of sides are either equal or supplementary.* Geometrically this becomes obvious by considering the triangle ABC′ instead of the triangle ABC, where $C'B = BC$. The triangles ABC′, ABC have the same area and the theorem applies to the pair of triangles (AC′B, DEF) as well as it does to the pair (ABC, DEF).

Theorem 24.

Areas of similar triangles are proportional to the squares on their corresponding sides.



This theorem follows at once from the previous one,

for $\frac{\triangle ABC}{\triangle DEF} = \frac{AB.BC}{DE.EF} = \frac{BC^2}{EF^2}$, since $\triangle^s$ ABC, DEF are similar

Otherwise, we may write

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} BC.AX}{\frac{1}{2} EF \cdot DY} = \frac{BC}{EF} \cdot \frac{AX}{DY} = \frac{BC^2}{EF^2},$$

since $\triangle^s$ ABX, DEY are similar and the ratio of their altitudes is equal to the ratio of their sides, *viz.*,

$$AX : DY = BC : EF.$$

Note. The areas are also proportional to the squares on other corresponding elements such as altitudes.

Example

In a triangle ABC, AB = 9 *cm*,, BC = 15 *cm*., CA = 12 *cm*., *and* AX *is drawn perpendicular to* BC. *Find the areas of the triangles* ABX *and* AXC.

(Refer to the diagram of Theorem 22)

$\because 15^2 = 9^2 + 12^2$, $\angle A = 90°$ and BC is the hypotenuse.

$\therefore AB^2 = BX.BC.$ $\therefore BX = 9 \times 9/15 = 5{\cdot}4$ cm.

$\because$ ABX : $\triangle$ ABC = $AB^2 : BC^2$, AB and BC being corres ponding sides.

$\therefore \triangle ABX : \triangle ABC = 9 \times 9 : 15 \times 15 = 9 : 25.$

$\therefore \triangle ABX = (9/25).\ \triangle ABC = (9/25)\ \frac{1}{2}.\ 9{\cdot}12 = 19{\cdot}44$ sq. cm.

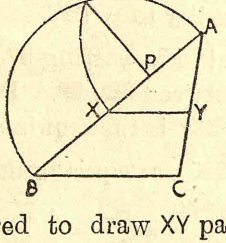
Similarly $\triangle ACX : \triangle ABC = AC^2 : BC^2 = 12 \times 12 : 15 \times 15$

$= 16 : 25.$

$\therefore \triangle ACX = (16/25).\ \triangle ABC = (16/25)\ \frac{1}{2}.\ 9{\cdot}12 = 34{\cdot}56$ sq. cm.

Construction 6.

To cut off a given fraction of a triangle by a line drawn parallel to a side.



Let it be required to draw XY parallel to BC so that the area of $\triangle$ AXY may be $(1/n)$th of that of $\triangle$ ABC.

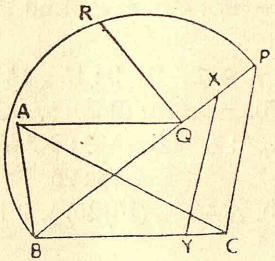
Take P in AB so that $AP = (1/n)$ AB. (P lies outside AB if $n < 1$.) Find the mean proportional AX between AP and AB. (This is done in the diagram by drawing PM perpendicular to AB to meet the semi-circle on AB as diameter at M and taking AX equal to AM.) Draw XY parallel to BC.

Proof. $\triangle^s$ AXY, ABC are equiangular and similar.

$$\therefore \frac{\triangle AXY}{\triangle ABC} = \frac{AX^2}{AB^2} = \frac{AP.AB}{AB^2} = \frac{AP}{AB} = \frac{1}{n}.$$

Construction 7

To draw a triangle similar to a given triangle and equal in area to another given triangle.



Suppose it is required to draw a triangle equal to the triangle ABC. On the base BC draw the triangle BPC similar to the other given triangle by making them equiangular. We have now to draw a triangle equal to $\triangle$ ABC in area and similar to $\triangle$ PBC.

Draw AQ parallel to BC cutting BP in Q. Find BX the mean proportional between BQ, BP. Draw XY parallel to PC cutting BC in Y. BXY is the required triangle.

Proof. $\triangle^s$ BXY, BPC are equiangular and similar.

$$\therefore \frac{\triangle BXY}{\triangle BPC} = \frac{BX^2}{BP^2} = \frac{BQ.BP}{BP^2} = \frac{BQ}{BP} = \frac{\triangle BQC}{\triangle BPC},$$ since BQC, BPC are triangles having equal altitudes, their bases being BQ, BP. But $\triangle$ BQC and $\triangle$ ABC are of equal area since AQ is parallel to BC.

$$\therefore \triangle BXY : \triangle BPC = \triangle BQC : \triangle BPC = \triangle ABC : \triangle BPC.$$

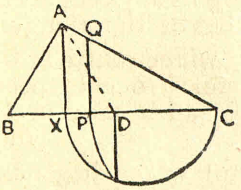
$$\therefore \triangle BXY = \triangle ABC.$$

Notes. (1) The triangle ABC may be given by its area or by its sides and angles. For example, we may have to draw an equilateral triangle of area 3·2 sq. in. We know then that $\triangle$ BPC is equilateral and that the area of $\triangle$ ABC $= 3{\cdot}2$ sq. in. We can start with any triangle ABC of area 3·2 sq. in. The simplest case would be to take $\angle B = 90°$, BC $= 3{\cdot}2''$, AB $= 2{\cdot}0''$, so that $\triangle$ ABC $= \frac{1}{2}$ AB.BC $= 3{\cdot}2$ sq. in. The rest of the construction follows as before.

(2) We may employ this construction to draw a polygon of given shape and of given area. Thus suppose for a simple example, we have to construct a regular pentagon of area 6 sq. in. If the extremeties of any side AB are joined to the centre O, then $\triangle$ OAB $= 6/5 = 1{\cdot}2$ sq. in. in area and its angles are $\angle O = 72°$, $\angle A = 54° = \angle B$.

We start the construction by drawing a right-angled traingle of area 1·2 sq. in. (say by taking the sides about the right angle equal to 2 and 1·2 inches) and converting it into a triangle of equal area and containing angles 54°, 54°, 72°. When $\triangle$ OAB is drawn, we can describe the circum-circle and complete the figure by stepping off chords equal to AB all round the circle.

(3) The construction may be adapted *to draw a line perpendicular to the base of a triangle so as to bisect the area.* Let ABC be a triangle which is to be bisected by a line perpendicular to BC. Let D be the mid-point of the base and X the foot of the perpendicular from A to the base, which we shall suppose to lie in the segment BD. We are required to get a triangle similar to $\triangle$ AXC but whose area equals that of $\triangle$ ADC. If CP is the mean proportional to CD, CX then PQ, drawn parallel to AX to cut AC at Q, is the required line.



Proof. $\triangle^{s}$ QPC, AXC are similar.

$$\therefore \frac{\triangle QPC}{\triangle AXC} = \frac{CP^2}{CX^2} = \frac{CD \cdot CX}{CX^2} = \frac{CD}{CX}$$

$$= \frac{\triangle ADC}{\triangle AXC}$$

$$\therefore \triangle QPC = \triangle ADC = \tfrac{1}{2} \triangle ABC.$$

(4) The last result may be extended to the case of drawing a line through the vertex inclined to BC at a given angle θ and so as to cut off $(1/n)$th of the area. We have to take $CD = BC/n$ and draw AX so that $\angle AXC = \theta$ and complete the construction as in Note (3).

Exercise 7

(1) XY is drawn parallel to the side BC of a triangle ABC cutting AB and AC in X and Y respectively.

AB = 5·4 cm., BC = 7·2 cm., B = 90°.

(i) If AX = 4·5 cm., calculate XY and AY.
(ii) If XY = 5·4 cm., calculate AX and AY.
(iii) If AY = 6·0 cm., calculate AX and AY.
(iv) If AY = 4·0 cm., calculate the ratio $\triangle AXY : \triangle ABC$

(2) AB and CD are chords of a circle, intersecting each other at O. If AO = 1·8 cm., CO = 2·4 cm., OB = 3·0 cm. and BD = 2·5 cm., calculate AC, OD and $\triangle AOC : \triangle BOD$.

(3) Points D, E, F, are taken on the sides BC, CA, AB respectively of a $\triangle ABC$ such that BD = DC, CE = 2EA, AF = 3FB. Find the ratio of the areas of the triangles DEF and ABC. (m)

(4) Construct a triangle ABC having BC = 6·8 cm., CA = 5·2 cm., and AB = 4·4 cm. Trisect the area by lines drawn perpendicular to BC.

(5) Construct a regular pentagon of area 7 sq. in.

(6) Construct a right-angled isosceles triangle ABC with its hypotenuse equal to 3·4″ and construct an equilateral triangle having the same area as $\triangle ABC$.

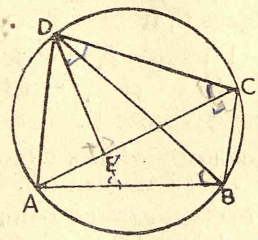
(7) Describe an equilateral triangle on a base of 2·3″ and construct a rectangle having its adjacent sides in the ratio 2 : 3 and its area equal to that of the triangle.

(8) From any point Q on a circle whose centre is C, QS is drawn perpendicular to a diameter PR so as to meet it at S. The tangents to the circle at P and Q meet in T. Prove that $\triangle PTQ : \triangle CRQ = PS : SR$ (m)

(9) ABCD is a cyclic quadrilateral whose diagonals intersect at O. Show that AB.BC : CD.DA = BO:OD. (m).

Theorem 25.

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of its opposite sides.



If ABCD be a cyclic quadrilateral, it is required to prove that AB.CD + AD.BC = AC.BD.

Draw DE to meet AC in E, so that $\angle ADE = \angle BDC$. (If BD bisects the angle ADC, E coincides with the point of intersection of the diagonals.)

In the triangles ADE and BDC,

$\angle ADE = \angle BDC$ by construction,

$\angle DAE = \angle DBC$ standing on the same arc DC,

and $\therefore \angle AED = \angle DCB$ and the triangles are similar.

$\therefore$ AD : BD = AE : BC = DE : DC.

$\therefore$ AD.BC = BD.AE. (i)

Again adding $\angle EDB$ to the equal angles ADE, BDC,

$\angle ADB = \angle EDC$,

$\angle ABD = \angle DEC$ on the same arc AD.

$\therefore \angle BAD = \angle CED$ and $\triangle^s$ ABD, CDE are similar.

$\therefore$ AB : EC = BD : CD = AD : DE.

$\therefore$ AB.CD = BD.EC. (ii)

$\therefore$ AD.BC + AB.CD = BD.AE + BD.EC = BD (AE + EC) = BD.AC.

Notes. (1) This is called Ptolemy's theorem and the result may also be deduced as follows :

If AB, BC, CD, DA, AC, BD are equal to a, b, c, d, x, y respectively,

$x^2 = a^2 + b^2 - 2ac \cos B$, from $\triangle$ ABC,
using the *cosine formula* (Th. 8).

$x^2 = c^2 + d^2 - 2cd \cos D$, from $\triangle$ ACD,
$y^2 = a^2 + d^2 - 2ad \cos A$, from $\triangle$ ABD,
$y^2 = b^2 + c^2 - 2bc \cos C$, from $\triangle$ BCD.

$$\therefore x^2\left(\frac{1}{ab} + \frac{1}{cd}\right) = \frac{a^2 + b^2}{ab} + \frac{c^2 + d^2}{cd}.$$

$\because \cos B = \cos(180° - D) = -\cos D.$

$$\therefore x^2 (ab + cd) = cd (a^2 + b^2) + ab (c^2 + d^2)$$
$$= (bc + ad)(ac + bd).$$

$$\therefore x^2 = (bc + ad)(ac + bd)/(ab + cd), \text{ and similarly}$$
$$y^2 = (ab + cd)(ac + bd)/(bc + ad).$$

$$\therefore x^2 y^2 = (ac + bd)^2 \quad \text{or} \quad xy = ac + bd,$$

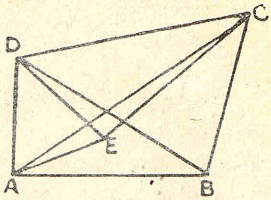
or AC.BD = AB.CD + AD.BC.

This method also gives the lengths of the diagonals, a remarkable result given by *Brahmagupta*, a brilliant Indian mathematician of the seventh century.

We further see that $x : y = (bc + ad) : (ab + cd)$
or AC : BD = (AB.AD + BC.CD) : (AB.BC + CD.AD)

See also Exercise 1 (3)

(2) The corresponding result for a non-cyclic quadrilateral may be found thus:



Draw DE and AE so as to intersect at E and such that $\angle ADE = \angle BDC$ and $\angle DAE = \angle DBC$. Join EC.

By construction $\triangle^s$ ADE and BDC are equiangular and similar. $\therefore$ AD : BD = DE : DC = AE : BC.

$$\therefore AD.BC = BD.AE. \qquad (i)$$

Again, in the triangles ABD and CED,

$\angle ADB = \angle EDC$, and the ratio of the sides about these angles are equal, for $AD : BD = DE : DC$ as proved above.

$\therefore \triangle^s ABD$ and CED are similar. (Th. 20)

$\therefore AB : EC = BD : CD = AD : DE.$

$\therefore AB.CD = BD.EC.$ (ii)

$\therefore$ By addition, $AD.BC + AB.CD = BD\,(AE + EC) > BD.AC.$

(Note that this method holds good for a cyclic quadrilateral also)

The result may be stated thus : *The sum of the rectangles contained by the pairs of opposite sides of a non-cyclic quadrilateral is greater than the rectangle contained by its diagonals.*

(3) The above results may be combined into a single theorem : *The rectangle contained by the diagonals of a quadrilateral does not exceed the sum of the rectangles contained by pairs of its opposite sides.*

Exercise 8

(1) P is a point in the arc BC of the circum-circle of an equilateral triangle ABC. Prove that $PA = PB + PC$.

(2) If R is the radius of the circum-circle of a triangle ABC and BD, CD are perpendicular to AB, AC, prove that

$$AC.BD + AB.CD = 2R.BC.$$

(3) ABCD is a cyclic quadrilateral and BD bisects the angle ABC. A and C are fixed points on the circle but B is variable in position. Show that $AB + BC$ varies as BD. (m)

(4) ABCD is a cyclic quadrilateral. If AE is a chord parallel to BD show that $AB.BC + AD.DC = BD.CE$. (m)

(5) ABCD is a parallelogram. The circle ABC cuts AD in E. Prove that the difference between AB^2 and AC^2 is equal to AD. AE. (m).

(6) ABC is a triangle and BCD is an equilateral triangle such that D and A are on opposite sides of BC. If AD meets the circle BCD in P, show that

$$AD = AP + BP + CP.$$ (m).

(5) ABCDE is a regular pentagon inscribed in a circle and P is any point on the arc AE. Prove that

$$PA + PC + PE = PB + PD.$$

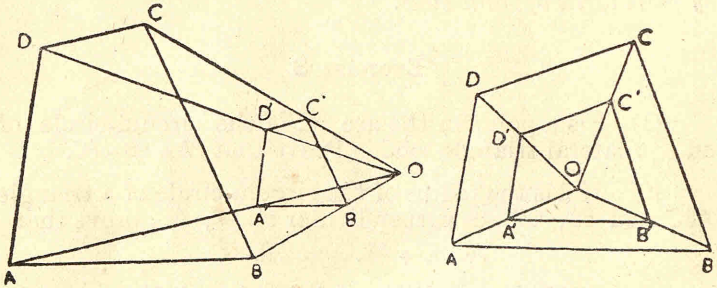
(6) If A, B, C, D are four consecutive vertices of a regular heptagon, prove that $AD^2 = AC\ (AB + AC)$. *(m)*

(7) ABCD is a quadrilateral in which AB, BC, CD, DA and AC are respectively equal to $(m^2 - n^2)\ (l^2 + 1)$, $2mn\ (l^2 + 1)$, $2l\ (m^2 + n^2)$, $(l^2 - 1)\ (m^2 + n^2)$ and $(m^2 + n^2)\ (l^2 + 1)$. Show that the quadrilateral is cyclic and hence deduce that $BD = 2\ (lm - n)\ (ln + m)$.

3. Similar Figures.

Theorem 26.

If two similar figures are similarly placed, the straight lines joining corresponding points meet in a point.



Let ABCD, A′B′C′D′ be to two similar figures similarly placed, *i.e.*, so placed that the corresponding sides are parallel. Suppose AA′, BB′ meet in O. Join CO, C′O.

∵ AB, A′B′ are parallel, $\triangle^s$ AOB, A′OB′ are equiangular.

∴ OB : OB′ = AB : A′B′ = BC : B′C′, ∴ ABCD, A′B′C′D′ are similar.

In $\triangle^s$ OBC and OB′C′, $\angle OBC = \angle OB'C'$, (∵ BC and B′C′ are parallel,) and OB : OB′ = BC : B′C′.

∴ Those triangles are similar.

∴ $\angle BOC = \angle B'OC'$. ∴ CC′ passes through O.

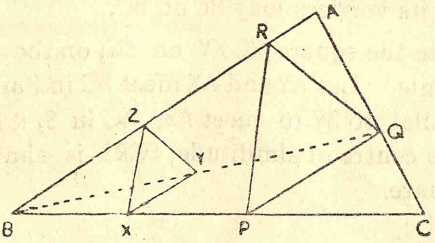
Similarly DD′ also passes through O.

Notes. (1) The diagrams here give the case of directly similar unequal figures. The result is also true of inversely similar figures. (Examine the converse.)

(2) O is called the **centre of similarity**, the **centre of similitude** or the **homothetic centre** of the two figures. (When equal figures are similarly placed O goes to infinity.) This property of similar figures can be applied to inscribe required figures in given figures as in the following examples.

Construction 8

To inscribe in a given triangle, a triangle similar to another given triangle.

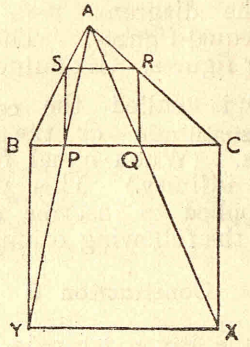


Suppose it is required to inscribe in a given $\triangle$ ABC, a triangle whose angles are α°, β°, γ° $(\alpha+\beta+\gamma=180)$, so that one of the sides is inclined at θ° to BC.

From any point X in BC draw XZ inclined at θ° to BC and cutting AB in Z. Complete $\triangle$ XYZ so as to have α°, β°, γ° for its angles. Join BY and produce it, if necessary, to meet AC in Q. Draw QR parallel to ZY to cut AB in R and QP parallel to XY to cut BC in P. PQR is then the required triangle. It is similar to $\triangle$ XYZ and the triangles have B for their homothetic centre.

Construction 9.

To inscribe a square in a given triangle so that two of its vertices may lie on a particular side.

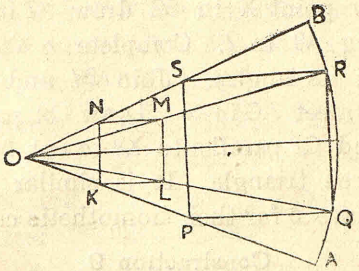


Let it be required to inscribe a square in △ ABC so that two of its vertices may lie on BC.

Describe the square BC XY on BC, on the side opposite to the triangle. Let AY and AX meet BC in P and Q. Draw PS, QR parallel to BY to meet AB, AC in S, R respectively. A being the centre of similitude, PQRS is similar to BCXY and is a square.

Construction 10

To inscribe a square in a given sector, so that two of its vertices may lie on the circumference.



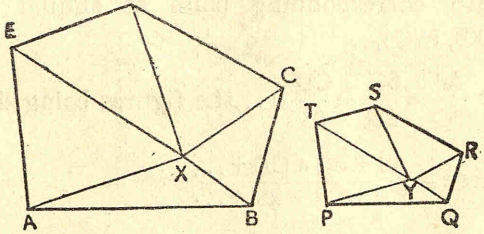
Let AOB be a sector in which we have to inscribe a square. Take K and N in OA and OB so that OK = ON. (KN will

be bisected at right angles by the central radius of the sector.) On KN describe the square KLMN. Let OL, OM meet the arc at Q, R. when produced, if necessary. Draw RS, QP parallel to KL to cut OB, OA at S, P. Then PQRS and KLMN which have O for their centre of similarity are both squares, the former being the required figure.

The square described on AB may be employed instead of the square described on KN.

Theorem 27.

Similar figures may be divided into the same number of similar triangles by joining the corresponding vertices to two corresponding points.



Let ABCDE, PQRST be two similar figures and X any point within the former. Determine the corresponding point Y within the latter so that $\angle YPQ = \angle XAB$ and $\angle YQP = \angle XBA$. Join X and Y to the vertices of the figures.

By construction $\triangle^s$ AXB, PYQ are equiangular.

$\therefore XB : YQ = AB : PQ = BC : QR$ and $\angle ABX = \angle PQY$.

$\because$ the figures are similar, $\angle ABC = \angle PQR$. $\therefore \angle XBC = \angle YQR$.

$\therefore \triangle^s$ BXC and QYR are similar. (Th. 20)

By continuing the argument similarly we can show that the successive pairs of triangles (CXD,RYS), (DXE,SYT) .... are all similar. The number of such pairs of similar triangles is equal to the number of sides of either figure.

Notes. (1) $\triangle^s$ AXB, BXC are *not necessarily* similar.

(2) X and Y may coincide with corresponding vertices of the figures.

Theorem 28.

The perimeters of similar polygons are proportional to the corresponding sides and their areas are proportional to the squares on the corresponding sides.

(Refer to the previous diagram.)

Let ABCDE, PQRST be two similar polygons with X, Y two corresponding points in them. Then by joining X, Y to the vertices of the figures, we can divide the figures into corresponding pairs of similar triangles such as (AXB, PYQ).

(i) $\because \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \ldots$ the figures being similar,

each of these $= \frac{AB + BC + CD + \ldots}{PQ + QR + RS + \ldots}$.

$\therefore$ The ratio of the perimeters of the figures is equal to the ratio of the corresponding sides.

(ii) $\because$ the pairs of triangles are similar,

$$\frac{\triangle AXB}{\triangle PYQ} = \frac{AB^2}{PQ^2}, \frac{\triangle BXC}{\triangle QYR} = \frac{BC^2}{QR^2} = \frac{AB^2}{PQ^2}, \frac{\triangle CXD}{\triangle RSY} = \frac{CD^2}{RS^2} = \frac{AB^2}{PQ^2}, \ldots$$

$$\therefore \frac{AB^2}{PQ^2} = \frac{\triangle AXB}{\triangle PYQ} = \frac{\triangle BXC}{\triangle QYR} = \frac{\triangle CXD}{\triangle RYS} = \ldots$$

$$= \frac{\triangle AXB + \triangle BXC + \triangle CXD + \ldots}{\triangle PYQ + \triangle QYR + \triangle RYS + \ldots}$$

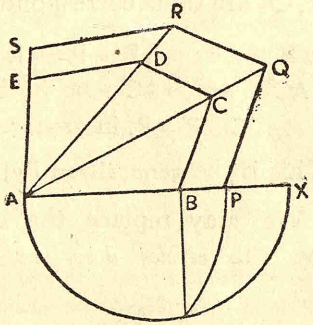
$$= \frac{\text{area of the polygon ABCDE}}{\text{area of the polygon PQRST}}.$$

Note. Two circles may be considered to be the limiting case of two similar rectilineal figures — for example, of regular polygons of the same number of sides—when the length of each side diminishes indefinitely and the number of sides increases indefinitely. Hence we find the result that *the circumferences of two circles are proportional to their radii and their areas are proportional to the squares on the radii.*

(X and Y may be taken at the centres.)

Construction 11

To construct a polygon similar to a given polygon and having n times its area.



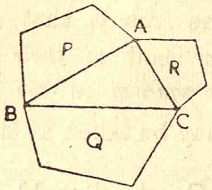
Let ABCDE.....be the given polygon. In AB take the point X so that $AX = n$. AB. Determine AP the mean proportional between AB and AX. Join AC, AD, AE,.....Draw PQ parallel to BC to cut AC in Q, and then draw QR parallel to CD to cut AD in R and proceed similarly. Complete the polygon APQRS.....similar to ABCDE.....

Proof. By construction the figures are similar.

$$\therefore \frac{\text{The area of APQRS....}}{\text{The area of ABCDE....}} = \frac{AP^2}{AB^2} = \frac{AB.AX}{AB^2} = \frac{AX}{AB} = \frac{n}{1}.$$

Theorem 29.

In a right-angled triangle any rectilineal figure described on the hypotenuse is equal to the sum of two similar and similarly described figures on the other sides.



Let P, Q, R be three similar figures, similarly described on the sides AB, BC, CA of $\triangle$ ABC right-angled at A, so that AB, BC, CA are their corresponding sides.

Then $\frac{Q}{BC^2} = \frac{R}{AC^2} = \frac{P}{AB^2} = \frac{R+P}{AC^2+AB^2} = \frac{R+P}{BC^2}$.

$\therefore Q = P + R$, in area.

Notes. (1) This is the generalised Pythagoras' theorem.

(2) We may replace the *similar figures* of the theorem by the *circles described on the sides as diameters.*

Example

An estate is represented on a plan by a qudrilateral ABCD *drawn to the scale* 1 *cm. to* 33 *yds. If* BD = 9·6 *cm., and the perpendiculars from* A *and* C *to* BD *measure* 3·2 *cm., and* 4·3 *cm. respectively, calculate the area of the estate in acres.*

(Draw your own diagram)

The area of a quadrilateral

$= \frac{1}{2}$ (any diagonal) (sum of the off-sets on it.)

$\therefore$ The area of the figure ABCD

$= \frac{1}{2} \times 9{\cdot}6\ (3{\cdot}2 + 4{\cdot}3) = 4{\cdot}8 \times 7{\cdot}5 = 36{\cdot}0$ sq. cm.

$\because$ 1 cm. represents 33 yds.,

1 sq. cm. represents 33×33 sq. yds.

$\therefore$ The area of the estate $= 36 \times 33 \times 33$ sq. yds,

$= 8{\cdot}1$ acres.

Exercise 9

(1) *If four straight lines are in proportion, and a pair of similar rectilineal figures are similar described on the first two lines, and also a pair of similar rectilineal figures are similarly described on the next two lines, then the four figures are in proportion.*

(2) *If three lines are in continued proportion and if similar figures are described on the first two lines, their areas are in the ratio of the first and the third lines.*

(3) ABCD is a trapezium whose diagonals intersect at O. If AO = 2 OC, find the ratio in which BD is divided at O and also the ratio $\triangle$ AOB : $\triangle$ COD. If OP, OQ drawn parallel to BC, CD, meet AB, AD in P, Q respectively, calculate the ratio APOQ : ABCD.

(4) A field in the form of a trapezium is represented by a diagram ABCD in which AB and CD are parallel and AB = 5·2 cm., B = 90°, BC = 2·5 cm., DC = 3·6 cm. If the side corresponding to BC measures 60 yds., find the length of the side corresponding to AB and also the area of the field.

(5) The map of a country whose boundary is made up of straight lines is a rectilineal figure in which 1″ represents 50 miles. If the area of the map is 7·2 sq. in., what is the area of the country?

(6) A field is represented by a diagram ABCD drawn to the scale 1″ to 66 yds. AC = 5·4″ and the perpendiculars to AC from B and D are 3·2″ and 1·8″ respectively. Find the area of the field in acres.

(7) Draw the triangle ABC having $\angle B = 30°$, BC = 7·2 cm., $\angle C = 80°$. Inscribe in it an equilateral triangle, a side of which is inclined at 20° to BC.

(8) In an equilateral triangle whose base measures 3″, inscribe a parallelogram whose sides are in the ratio 4 : 7, one of its angles being equal to 48°.

(9) Construct the triangle ABC having AB = 3·8″, BC = 2·9″, CA = 1·9″. Inscribe a rhombus with one of its angles equal to 72° in the triangle, so that one of its sides may lie along AB.

(10) Construct a triangle ABC having $\angle B = 40°$, BC = 7·5 cm., $\angle C = 65°$ and inscribe a square in it, with two of its vertices on AC.

(11) Describe the sector of a circle of radius 2·6″ and of angle 40°. Inscribe a square in it, with two of its vertices (i) on the arc, (ii) on a bounding radius.

(12) Inscribe two rectangles whose sides are in the ratio 5 : 3 in a sector of a circle of radius 2·8″, the angle at the centre being equal to 52°.

(13) Describe the segment of a circle of radius 4·6 cm., on a chord of length 7·2 cm. Inscribe a rectangle in the segment so that its sides may be in the ratio of 3 : 7 and one of the longer sides may lie along the chord.

(14) Inscribe a square in a regular hexagon on a base of 2″ so that two of its sides may be parallel to two sides of the hexagon.

(15) ABCD is a quadrilateral in which AB = 2·3″, A = 100°, B = 115°, AD = 1·8″, BC = 2·6″. Describe a similar quadrilateral of twice the area.

(16) Draw a semi-circle of radius 1·8 and describe another semi-circle of $2\frac{1}{2}$ times the area.

(17) Draw a circle of radius 4″ and divide it into five equal parts in area, by concentric circles.

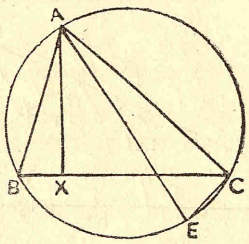
(18) Construct (without using a protractor) a regular pentagon on a base of 2.4″ and describe a similar figure of three-fourths that area.

4. Properties of Triangles.

Theorem 30.

√ done

The rectangle contained by any two sides of a triangle is equal to the rectangle contained by the altitiude drawn perpendicular to the third side and the circum-diameter.



Let AX be the altitude perpendicular to BC of a triangle ABC. Let AE be the diameter through A, of the circum-circle. Join EC.

$\angle AXB = \angle ACE = 90°$,

$\angle ABX = \angle AEC$, standing on the same arc AC.

$\therefore \angle BAX = \angle EAC$ and $\triangle^{s}$ ABX, AEC are equiangular.

$\therefore \angle AB : AX = AE : AC$ and $AB.AC = AX.AE$.

Notes. (1) Study what happens to the diagram and to the proof if $\angle B$ is a right angle or an obtuse angle. (The result is unaltered.)

(2) This may be called **Brahmagupta's theorem** in honour of one of our greatest mathematicians, who enunciated the result in 628 A.D.

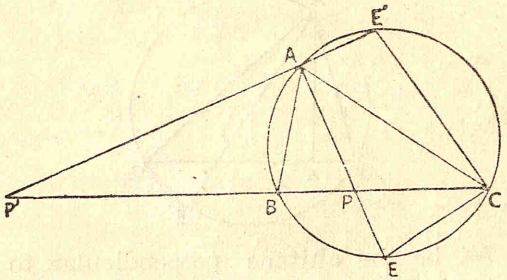
(3) Multiplying the result of the theorem by BC, we get $AB.BC.CA = BC.AX.AE$,
or since $\frac{1}{2}$ BC. AX $= \triangle$, the area of the triangle ABC, we may write this in symbols, $abc = 2\triangle \times 2R = 4R\triangle$, R being the circum-radius.

$\therefore R = abc/4\triangle = abc/4\sqrt{[s(s-a)(s-b)(s-c)]}$

giving *an expression for the circum-radius in terms of the sides.*

Theorem 31.

If the vertical angle of a triangle is bisected *internally* by a straight line which cuts the base, the rectangle contained by the sides of the triangle equals the rectangle contained by the segments of the base *together with* the square on the straight line bisecting the angle.



Let AP bisect ∠ BAC of a triangle ABC and when produced, meet the circum-circle again at E. Join EC. It is required to prove that

$$AB \cdot AC = BP \cdot PC + AP^2.$$

In the triangles APB and AEC,

∠ BAP = ∠ EAC, since AP bisects ∠ A.

∠ ABP = ∠ AEC, on the same arc AC.

∴ ∠ APB = ∠ ACE and the triangles are similar.

$$\therefore AB : AE = AP : AC.$$

$$\therefore AB \cdot AC = AP \cdot AE = AP\,(PE + AP) = AP \cdot PE + AP^2$$
$$= BP \cdot PC + AP^2, \text{ since BC, AE are chords through P.}$$

$$[\,AB \cdot AC = AP \cdot AE = AE(AE - EP) = AE^2 - AE \cdot EP = AE^2 - CE^2$$

$\because AE : CE = CE : EP$ by similar triangles AEC, PEC]

Notes (1) Instead of the internal bisector AP if we consider the external bisector AP', we see that the proof is unaltered after replacing P, E by P', E', except for the sign of AP^2 in the last three steps. In that case $AE' = P'E' - AP'$.

The resulting theorem may be stated thus: **If the vertical angle of a triangle is bisected *externally* by a straight line which meets the base, the rectangle contained by the sides equals the rectangle contained by the segments of the base *diminished by* the square on the line bisecting the vertical angle.**

(2) If E is the middle point of the minor arc BC, E′ is the mid-point of the major arc BC and *vice-versa*. Hence if AB < AC, E′ lies in the arc AC and if AB > AC, E′ lies in the arc AB. If the position of E′ is not properly marked, the proof becomes impossible.

(3) This theorem and the previous one give two different expressions for the rectangle contained by two sides of a triangle.

(4) This theorem enables us to calculate the length of the bisector of any angle of a triangle. Using the present diagram,

$$AB = c,\ BC = a,\ CA = b,\ BP : PC = c : b.$$

$$\therefore BP = ac/(b+c),\ PC = ab/(b+c).$$

$$AB.AC = BP.PC + AP^2.$$

$$\therefore AP^2 = AB.AC - BP.PC = bc - a^2bc/(b+c)^2$$

$$AP = \sqrt{\left[\ bc\ \left\{\ 1 - \frac{a^2}{(b+c)^2}\ \right\}\ \right]} \qquad \text{[See Ex. 5, (13)]}$$

$$\text{Similarly } AP' = \sqrt{\left[\ bc\ \left\{\ \frac{a^2}{(b\sim c)^2} - 1\ \right\}\ \right]}.$$

Exercise 10

(1) M is a point on BC and R is a point on the circum-circle of a triangle ABC, such that $\angle MAB = \angle RAC$ and AM and AR lie inside the angle BAC. Show that

$$AM.AR = AB.AC.$$

Deduce the results of Theorems 30 and 31.

(2) ABC is a triangle inscribed in a circle. AD is drawn touching the circle and meeting BC produced in D. Prove that the diameters of the circles about the triangles ABD and ACD are in the ratio of AD to CD.

(3) P is a point on the circum-circle of a cyclic quadrilateral ABCD and p_1, p_2, p_3, p_4, are its distances from the sides AB, BC, CD, DA respectively. The distances of P from the diagonals AC, BD are d_1, d_2 respectively. Prove that $d_1 d_2 = p_1 p_3 = p_2 p_4$.

(4) In a cyclic quadrilateral ABCD show that AC : BD = (AB.AD + CB.CD) : (BA.BC + DA.DC). Hence or otherwise deduce that the diagonals are given by

$$AC^2 = (ac + bd)(ad + bc)/(ab + cd)$$

and $$BD^2 = (ac + bd)(ab + cd)/(ad + bc),$$

a, b, c, d being the lengths of AB, BC, CD, DA respectively.

(5) ABCD is a cyclic quadrilateral whose diagonals intersect at O.

Show that AB.BC : CD.DA = BO : OD *(m)*

(6) Construct a triangle ABC given

(i) BC = 4·2 cm., AB.AC = 29·4 sq. cm., and A = 30°

(ii) BC = 2·3″, AB.AC = 5·4 sq. in., altitude = 1·8″

(iii) BC = 6·5 cm., AB.AC = 38·88 sq. cm., and the internal bisector AP = 5·4 cm.

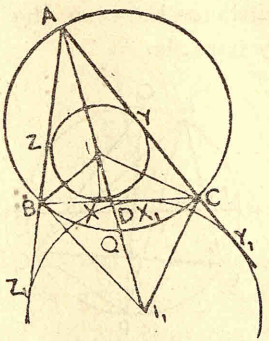
Special points associated with a triangle.

The student is already familiar with the property that the three perpendicular bisectors of the sides are concurrent at a point called the *circum-centre* and that the point is equidistant from the vertices.

The internal bisectors of the three angles are also concurrent, the point of concurrence being called the *in-centre*. Again any internal bisector and the external bisectors of the remaining angles are also concurrent, that point of concurrence being called the *ex-centre* opposite the vertex through which the internal bisector passes. [See Ex. 5, (11)]. Each of these four points, usually denoted by I, I_1, I_2, I_3, is equally distant from the sides of the triangle.

Examples

(1) *Some properties of the in-and the ex-circles.*



Let the in-circle touch BC, CA, AB at X, Y, Z and the ex-circle opposite to A touch these sides at X_1, Y_1, Z_1. Let AII_1 cut the circum-circle ABC at Q. AB, AC are the direct common tangents and BC a transverse common tangent to the in-circle and the ex-circle opposite to A.

$\because \angle BAQ = \angle QAC$, $\quad BQ = QC$.

$\angle BIQ = \angle IBA + \angle IAB = \frac{1}{2}(A + B)$,

$\angle IBQ = \angle IBC + \angle CBQ = \angle IBC + \angle CAQ = \frac{1}{2}(A + B)$.

$\therefore \angle BIQ = \angle IBQ$. $\quad \therefore IQ = BQ = QC$.

$\therefore$ Q is the centre of the circle BIC.

$\because$ BI, BI_1, bisect $\angle$ B internally and externally, $\angle IBI_1 = 90°$.

$\therefore$ II_1 is the diameter of the circle through I, B, I_1 which also passes through C and its centre Q is the mid-point of II_1 and lies on the circum-circle.

Similarly the middle point of every one of the st. lines joining any two of the in-and the ex-centres lies on the circum-circle.

Again $BX = BZ$, $CX = CY$, $AY = AZ$; $\quad \therefore BX + CY + AY = s$, where s is the semi-perimeter of the triangle $= \frac{1}{2}(a + b + c)$.

$\therefore BX = BZ = s - b$, $\quad CX = CY = s - c$, $\quad AY = AZ = s - a$.

Further $AZ_1 = AB + BZ_1 = AB + BX_1$, $\quad AY_1 = AC + CY_1 = AC + CX_1$.

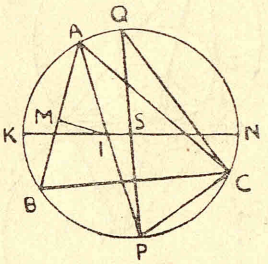
$\therefore AZ_1 = AY_1 = \frac{1}{2}(AZ_1 + AY_1) = s$; $\quad CX_1 = s - b$, $BX_1 = s - c$.

$\therefore BX = CX_1$ and XX_1 has the same mid-point D as BC.

$\triangle ABC = \triangle BIC + \triangle CIA + \triangle AIB = \frac{1}{2} r(a + b + c) = rs$.

Similarly $\triangle ABC = r_1(s - a)$.

(2) Find the distance between the circum-centre and the in-centre of any triangle.



Let S, I, be the circum-centre and in-centre of a triangle ABC and let SI produced cut the circum-circle at K and N. Let AI meet the circle again at P and PS meet the circle at Q. Draw IM perpendicular to AB.

Then $R^2 - SI^2 = (R - SI)(R + SI) = KI.IN = AI.IP = AI.PC$, since P is the centre of $\odot$ BIC as proved before.

In the triangles AMI and PCQ.

$\angle AMI = 90° = \angle PCQ$ and $\angle MAI = \angle PAC = \angle PQC$.

$\therefore$ The triangles are equiangular and $MI : PC = AI : PQ$.

$\therefore AI.PC = MI.PQ = r.2R = 2Rr$,

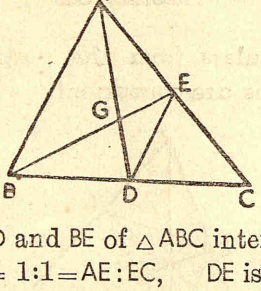
r being the in-radius and R the circum-radius.

$\therefore R^2 - SI^2 = 2Rr$, or $SI^2 = R^2 - 2Rr = R(R - 2r)$, which shows that *in any triangle the circum-radius is never less than the diameter of the in-circle.*

Notes. (1) This theorem was given by **Euler** (1707-83), one of the greatest mathematicians of the world, who was partially or totally blind during most of his life.

(2) The student is advised to proceed similarly and prove that $SI_1^2 = R^2 + 2Rr_1$, where I_1, is the centre of the ex-circle opposite to A and r_1, its radius.

(1) Show that the three medians of a triangle are concurrent at a point at which each median is trisected.



Let the medians AD and BE of $\triangle ABC$ intersect in G. Join DE.

$\because$ BD : DC = 1 : 1 = AE : EC, DE is parallel to AB.

$\triangle^s$ ABG and GDE are equiangular.

$\therefore \frac{AG}{GD} = \frac{BG}{GE} = \frac{AB}{DE} = \frac{AC}{EC} = \frac{2}{1}$, from similar $\angle^s$ ABC and CED.

$\therefore$ The median through B cuts the median through A at G which is a point of trisection of both. Similarly if the median through C cuts the median through A at G_1, then

$$AB_1 : G_1D = 2 : 1 = AG : GD.$$

$\therefore$ G_1 coincides with G, and the three medians meet at G, a point of trisection of each. (G is called the *centroid.)*

(4) *Given the base and the vertical angle of a triangle, find the locus of its centroid.*

Let G be the centroid of $\triangle$ ABC and D the mid-point of the base BC. Draw GK, GL, respectively parallel to AB, AC, so as to meet the base in K and L.

(GKD, ABD) and (GDL, ADC) are pairs of similar triangles.

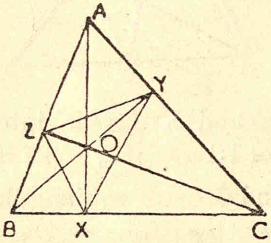
$\therefore$ DK : DB = DG : DA = 1 : 3, and DL : DC = DG : DA = 1 : 3.

$\therefore$ K and L are fixed points, being the points of trisection of BC.

$\because \angle KGL = \angle A$, the locus of G is a segment of a circle on KL containing the angle A.

Theorem 32.

The perpendiculars from the vertices of any triangle to the opposite sides are concurrent.



Let AX and BY be the perpendiculars from A and B on the opposite sides and let them intersect at O. Join CO and produce it to meet AB in Z. Join XY.

Since AXB, AYB are rt. angles, A, Y, X, B are concyclic.

$\therefore \angle BAX = \angle BYX$ on the same arc BX.

Again OXC and OYC being right angles O, X, C and Y are concyclic.

$\therefore \angle XOC = \angle XYC$, on the same arc XC.

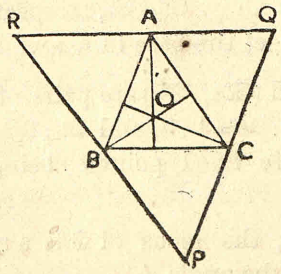
$\therefore \angle AOZ = \angle XYC$ and $\angle ZAO = \angle BYX$.

$\therefore \angle AOZ + \angle ZAO = \angle XYC + \angle BYX = 90°$.

$\therefore \angle AZO = 90°$, or CZ is perpendicular to AB.

$\therefore$ The three altitudes AX, BY, CZ are concurrent at O.

Alternative proof.



Draw parallels to the sides of the triangle through the opposite vertices so as to form the triangle PQR.

$\because$ ARBC, ABCQ are both parallelograms, $RA = BC = AQ$.

$\therefore$ A is the middle point of RQ and similarly B and C are middle points of RP and PQ respectively,

Also since BC, RQ are parallel, the perpendicular to BC is also perpendicular to RQ.

$\therefore$ The altitudes of the triangle ABC are the perpendicular bisectors of the sides of $\triangle$ PQR, and are therefore concurrent at the circum-centre of $\triangle$ PQR.

Notes. (1) O is called the orthocentre of $\triangle$ ABC.

(2) XYZ, the triangle formed by the feet of the altitudes, is called the pedal triangle with respect to $\triangle$ ABC.

(3) In the former figure of the theorem, ABXY is a cyclic quadrilateral. $\therefore \angle YXC = \angle BAC$ and similarly $\angle ZXB = \angle BAC$. Since each of these angles YXC, ZXB is equal to $\angle A$, BC bisects externally the angle X of the pedal triangle. Similarly CA and AB bisect the angles Y and Z of the pedal triangle.

Again each of the angles AXY, AXZ is equal to $90° - A$. Hence the altitudes bisect internally the angles of the pedal triangle. *The angles of the pedal triangle are*

$$180° - 2A, \quad 180° - 2B, \quad 180° - 2C.$$

To calculate the sides, we note that the triangles AZY, BXZ, CYX are all equiangular with $\triangle$ ABC. For example, in the triangles AZY and ABC,

$\angle A$ is common, $\angle AZY = \angle ACB$ and $\angle AYZ = \angle ABC$.

$\therefore \triangle^s$ AZY, ABC are similar and $YZ : BC = ZA : AC$.

$ZA/AC = \cos A$; $\quad \therefore YZ = a \cos A$

and similarly $ZX = \cos B$, $\quad XS = c \cos C$.

Putting $a = 2R \sin A$,...the magnitudes of the sides of the pedal triangle may also be written in the form

$$YZ = R \sin 2A, \quad ZX = R \sin 2B, \quad XY = R \sin 2C.$$

If R′ is the circum-radius of the pedal triangle,

$YZ = 2R'.\ \sin YXZ$, or. $R \sin 2A = 2R' \sin (180° - 2A)$.

$\therefore R' = \frac{1}{2}R$; *i.e.*, the circum-radius of the pedal triangle is half the circum-radius of the original triangle.

(4) The above results are obtained on the assumption that ABC is an acute-angled triangle, as in the diagram. If A is a right angle, it is easily seen from a diagram that the altitudes are concurrent at A, which is therefore the orthocentre. The pedal triangle shrinks into the altitune through A.

(5) If $\angle A$ is obtuse, the perpendiculars from B and C to the opposite sides lie outside the triangle and also cut each other outside. Hence in this case, the orthocentre is outside the triangle. (The student must draw a separate figure to illustrate this.) In fact from the diagram of the theorem we find that, of the four points A, B, C, O which are the vertices and the orthocentre of a triangle, each is the orthocentre of the triangle formed by the other three points. It will be further seen that the pedal triangle is the same for the triangles formed with the points of an orthocentric tetrad, the last phrase "orthocentric tetrad" denoting four points, each of which is the orthocentre of the triangle formed by the other three.

From the diagram of the theorem, we note that O, X, C, Y are concyclic.

$\therefore \angle OXY = \angle OCY = 90° - \angle YOC = 90° - (180° - \angle BOC) = \angle BOC - 90°$; $\angle OXZ = \angle OBZ = 90° - \angle ZOB = 90° - 180° + \angle BOC = \angle BOC - 90°$. $\therefore \angle YXZ = 2\angle BOC - 180°$.

Again $\angle OYX = \angle OCB$ and $\angle OYZ = \angle OAZ = \angle OCB$.

$\therefore \angle XYZ = 2\angle OCB$ and similarly $\angle XZY = 2\angle OBC$.

From this we infer that the angles of the pedal triangle are $2A - 180°$, $2B$, $2C$, if the angle A of the original triangle ABC is obtuse. The sides in this case are $-a \cos A$, $b \cos B$, $c \cos C$. (cos A is negative.)

(6) The vertices and the orthocentre are the in-and the ex-centres of the pedal triangle. When

$\triangle$ ABC is acute-angled, O is the in-centre of the pedal triangle and the vertices are the ex-centres. When ABC is an obtuse-angled triangle, the vertex at the obtuse angle is the in-centre and the other vertices and the orthocentre are the ex-centres.

(7) With reference to the pedal triangle, the original triangle (when acute-angled) is the ex-central triangle. By changing the letters of the diagram (A, B, C) into (I_1, I_2, I_3)(X, Y, Z,) into (A, B, C), and O into I, we see that,

(i) The in-centre and the ex-centres of any triangle form an ortho-centric tetrad.

(ii) The following groups of points are collinear:
(A, I, I_1), (B, I, I_2), (C, I, I_3), (I_1, C, I_2), (I_2, A, I_3,), (I_3, B, I_1).

(iii) The angles of the ex-central triangle are
$90° - \frac{1}{2}A$, $90° - \frac{1}{2}B$, $90° - \frac{1}{2}C$.

(iv) The triangles I_1BC, I_2CA, I_3AB, $I_1I_2I_3$ are similar.

(v) The radius of the circum-circle of the ex-central triangle is double the circum-radius of the original triangle.

Draw a good diagram and verify these properties independently.

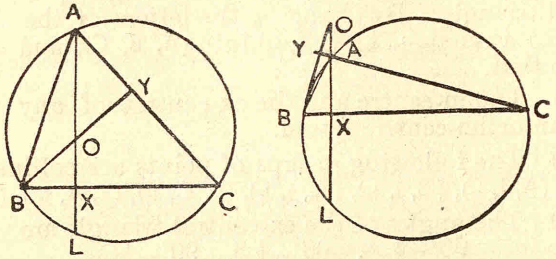
(8) Since AZOY is a cyclic quadrilateral, $\angle BOC = 180° - A$. Hence given BC and A, O lies on the segment of a circle on BC containing an angle of $180° - A$. This angle being the supplement of A, the arc BOC is exactly equal to the arc BC of the circum-circle ABC which does not contain A and in which BC subtends an angle $180° - A$ at any point. The remaining portion of the circle BOC in which BC subtends an angle equal to A at any point is equal to the arc BAC of the circum-circle. Hence the circle BOC is equal to the circle ABC. Similarly the circle COA, AOB are equal to the circle ABC.

This result is easily established from the trigonometrical theorem—*Any chord/sine of the angle subtended by the chord at the circumference = circum-diameter* or $a/\sin A = 2R$. [Ex. 6, (7)]

In the triangle BOC, $BC = a$, $\angle BOC = 180° - A$.
$\therefore$ The circum-diameter, $2R_1 = a/\sin(180° - A) = a/\sin A = 2R$
$\therefore R_1 = R$ and $\bigcirc BOC = \bigcirc ABC$.

Theorem 33.

If any altitude of a triangle is produced to meet the circum-circle again, the portion between that point of intersection and the orthocentre is bisected by the side perpendicular to the altitude.



Let AX, BY be the altitudes meeting at the orthocentre O. Let OA meet the circum-circle again at L.

In the triangles BXO and BXL, BX is common,

$$\angle BXO = 90^\circ = \angle BXL,$$

$\angle BOX = \angle BCA$, since the points O, X, C, Y are concyclic,

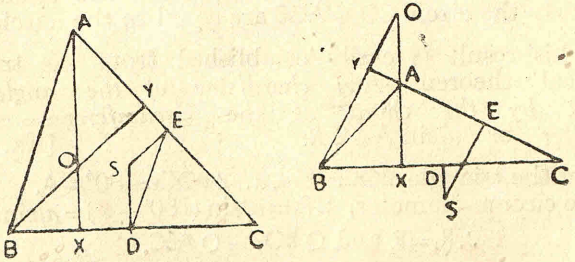
$= \angle BLX$ on the same arc AB of the circum-circle.

$\therefore$ The triangles are equal in all respects and $OX = XL$.

Note. If the altitude AX cuts the circum-circle at L, we have to take O in XA, so that $XO = LX$ to get the orthocentre.

Theorem 34.

The distance of the orthocentre of a triangle from any vertex is double the distance from the circum-centre to the opposite side.



Let the altitudes AX, BY intersect at O the orthocentre, and let the perpendicular bisectors of BC, CA meet in S the circum-centre. Join DE.

SD, AO, which are both perpendicular to BC, are parallel.

SE, BO, which are both perpendicular to CA, are also parallel.

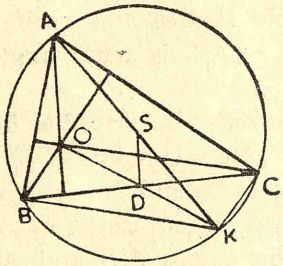
DE is parallel to AB, since D, E are mid-points of BC, CA.

$\therefore$ $\triangle^s$ SDE and OAB, having their corresponding sides parallel, are similar.

$\therefore$ SD : AO = SE : BO = DE : AB = 1 : 2, since DE = $\frac{1}{2}$ AB.

$\therefore$ SD = $\frac{1}{2}$ AO, SE = $\frac{1}{2}$ BO and smilarly SF = $\frac{1}{2}$ CO, where F is the mid-point of AB.

Alternative proof.



The altitudes AX, BY, CZ meet in O. S is the circum-centre and AK the diameter through A. Let OK meet BC in D. Join SD.

BO, KC, both being perpeudicular to AC, are parallel. CO, KB, both being perpendicular to AB, are also parallel.

$\therefore$ BOCK is a parallelogram and D is the middle point of both OK and BC.

S is the mid-point of AK and D is the mid-point of OK.

$\therefore$ SD is half of AO and parallel to it or perpendicular to BC.

Examples

(1) O *is the orthocentre of a triangle* ABC *and* AO, BO, CO *meet the circum-circle of the triangle again at* L, M, N, *respectively. Prove that* A, B, C *are the circum-centres of the triangles* OMN, ONL, OLM *respectively.*

Let AO, BO, CO meet BC, CA, AB at X, Y, Z, respectively.

$\angle ANO = \angle ABX$ ($\because$ A, N, B, C are concyclic)

$= \angle AON$ ($\because$ O, X, B, Z are concyclic)

$\therefore AN = AO.$

$\angle AOM = \angle ACX$ ($\because$ A, B, M, C are concyclic)

$= \angle AOM$ ($\because$ O, X, C, Y are concyclic)

$\therefore AM = AO.$

$\therefore AN = AO = AM$, and A is the circum-centre of $\triangle OMN$. Similarly B and C are the circum-centres of $\triangle ONL$ and $\triangle OLM$ respectively.

(Draw suitable figures and verify that this proof holds good when $\triangle ABC$ is acute-angled or when *any* of its angles is obtuse.)

[The result may also be proved as follows :

$\therefore$ Z, Y, are mid-points of ON, OM,

AB and AC bisect ON and OM at right angles.

$\therefore$ A is the circum-centre of $\triangle OMN$.]

$\because AN = AM$, the angles ALN and ALM subtended by the equal chords at L are either equal or supplementary.

$\therefore$ AL (or OL) bisects $\angle MLN$ interually or externally

Similarly OM, ON bisect $\angle NML$, $\angle LNM$ respectively.

$\therefore$ O *is the in-centre or an ex-centre of* $\triangle$ LMN *according as* $\triangle$ABC *is acute-angled or obtuse-angled.*

Note that O is the homothetic centre of $\triangle^s$ XYZ, LMN whose corresponding sides are parallel.

(2) *Construct the triangle* ABC *given* AS = 1·5″, OS = ·8″ AO = 1·2″.

(Refer to the diagram of Th. 34.)

Construct $\triangle AOS$ from the given data. Draw SD parallel to AO and equal to half of it. Draw the line perpendicular to SD at D and with centre S and radius equal to SA, draw an arc to cut that line at B and C. ABC is the required triangle.

Exercise 11

(1) If SO is equally inclined to AB and AC. show that $AO = AS$ and $\angle A = 60°$. (*m*)

(2) If $\angle A = 60°$, prove that the points S, O, I, I_1, B, C are concyclic.

(3) Prove that $AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 = 4R^2$. (*m*)

(4) If the points A, B, C, D are concyclic, show that the orthocentres of the triangles BCD, CDA, DAB, ABC lie on an equal circle.

(5) If the internal bisectors of the angles of a triangle ABC meet the circum-circle at P, Q, R show that the in-centre of the triangle ABC is the orthocentre of the triangle PQR. (*m*)

(6) The triangle formed by the circum-centres of the triangles AOB, BOC, COA is congruent with ABC and has O for its circum-centre and S for its orthocentre. (*m*)

(7) Deduce the result $AO = 2SD$ from the similar triangles PQR, ABC of the secoed diagram of Theorem 32.

(8) *If* X, Y, Z, *are the feet of the altitudes and* D *the mid-point of* BC, *show that* $R^2 - SO^2 = 2\ AO \cdot OX = 2\ BO \cdot OY = 2\ CO \cdot OZ = 2\ (BD^2 - OD^2)$ *and also deduce the result,*

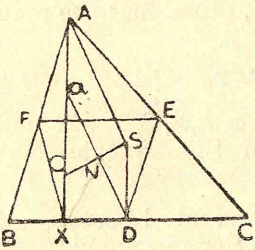
$$SO^2 = R^2 - 8R^2 \cos A \cos B \cos C.$$

(9) In any triangle ABC, the altitudes BY, CZ are drawn and the tangents at B and C to the circum-circle intersect at P. Show that AP bisects ZY.

(10) P, Q, R are points on the sides BC, CA, AB of a triangle ABC such that $\angle P = \angle A$, $\angle Q = \angle B$, $\angle R = \angle C$. Show that the circum-circles of the triangles ARQ, BPR, CQP pass through S, the orthocentre of the triangle PQR, which is also the circum-centre of the triangle ABC.

Theorem 35.

In any triangle, the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides and the middle points of the lines joining the orthocentre to the vertices, all lie on a circle, whose centre is the middle point of the line joining the orthocentre to the circum-centre and whose radius is half the circum-radius of the triangle.



Let D, E, F be the mid-points of the sides BC, CA, AB respectively, S the circum-centre, O the orthocentre, X the foot of the altitude AX and α the mid-point of OA. Join DE, EF, FX, α E, SD, SA, SO, Dα. Let SO and Dα cut at N.

(i) F is the mid-point of the hypotenuse AB of △ ABX.

$\therefore$ FX = FB = FA.

$\therefore$ ∠ FXB = ∠ FBX which is equal to ∠ FED, the opposite angle of the parallelogram BDEF.

$\because$ ∠ FXB = ∠ FED, F, X, D, E are concyclic, or X lies on ⊙ DEF.

Similarly Y and Z, the feet of the other altitudes, lie on ⊙ DEF.

(ii) Next consider the quadrilateral αEDX. Eα, the join of the mid-points of CA and OA, is parallel to CO and is hence perpendicular to AB or DE.

$\therefore$ ∠ αED = 90° and also ∠ αXD = 90°.

$\therefore$ α, X, D, E are concyclic.

$\therefore$ α lies on the circle through X, D, E which also passes through F, Y, Z.

Similarly β and γ the middle points of BO and CO also lie on that circle.

(iii) α being the middle point of AO, $SD = O\alpha = A\alpha$.

SD being equal and parallel to $O\alpha$, the triangles αON and NSD are equal in all respects.

$\therefore$ N is the mid-point of SO as well as of αD.

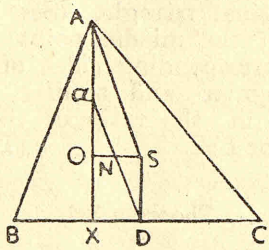
$\because \angle \alpha XD = 90°$, αD is a diameter of the circle through α, D, X, N the mid-point of αD, is its centre and it is also the mid-point of SO.

Again since $A\alpha$ and SD are equal and parallel,

αD and SA are equal and parallel.

$\therefore$ The diameter of the circle through the nine points is equal to the radius of the circum-circle. Its radius is half the circum-radius.

Alternative proof.



Consider the three representative points α, X, D specially connected with the vertex A. D is the mid-point of the side opposite to A, X is the foot of the perpendicular from A on the opposite side and α the mid-point of the line joining A to the orthocentre. Join AS, αD and OS. S and O being the circum-centre and the orthocentre. Let αD and OS cut each other at N.

SD is equal and parallel to $O\alpha$.

$\therefore$ The triangles α ON, NSD are equal in all respects and αD and OS bisect each other.

N is the middle point of the hypotenuse αD of the right-angled triangle αXD and it is the centre of its circum-circle.

Again since SD and $A\alpha$ are equal and parallel, $\alpha D = AS$.
$\therefore$ The radius of $O\ \alpha XD = \frac{1}{2}\ \alpha D = \frac{1}{2}\ AS = \frac{1}{2}$ circum-radius.

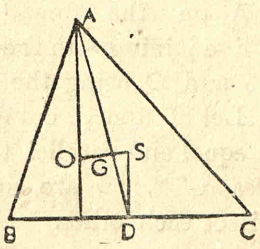
$\therefore$ The circle whose centre is mid-way between the orthocentre and the circum-centre and whose radius is half the circum-radius passes through the points α, X, D specially related to the vertex A. Similarly the circle also passes through the two groups of points (β, Y, E) and (γ, Z, F) which are specially related to the vertices B and C respectively.

Notes. (1) The circle DEF is called the **nine-point circle.** *It is the circum-circle of the pedal triangle* XYZ *or of the middle point triangle* DEF.

(2) It has been already proved that the circum-radius of the pedal triangle is half the circum-radius of the original triangle. (See Th. 32, Note 3.) Since the sides of the middle point triangle DEF are parallel to the corresponding sides of the triangle ABC, they are equiangular and similar. Therefore their circum-radii are in the ratio of their corresponding sides, *viz.*, DE : AB or 1 : 2. (Th. 20, Example 2.)

Theorem 36.

The centroid of a triangle lies on the line joining the circum-centre to the orthocentre and is a point of trisection of that line.



Let the line joining the orthocentre O to the circumcentre S of a triangle ABC intersect the median AD in G.

$\therefore$ SD is parallel to AO, the triangles AOG and SGD are equiangular. $\therefore$ OG : GS = AG : GD = AO : SD.

But AO = 2 SD; $\therefore$ AG = 2 GD and OG = 2 GS.

$\therefore$ AD = 3 GD, OS = 3 GS.

$\therefore$ G which is the point of trisection of AD nearer the base, is the centroid and it lies on SO. It is also the point of trisection of OS nearer to S.

Notes. (1) N the nine-point centre bisects and G the centroid trisects OS. NS = $\frac{1}{2}$ OS, GS = $\frac{1}{3}$ OS. $\therefore$ NG = $\frac{1}{6}$ OS.

(2) The line on which lie the orthocentre, the nine-point centre, the centroid and the circumcentre is called Euler line of the triangle.

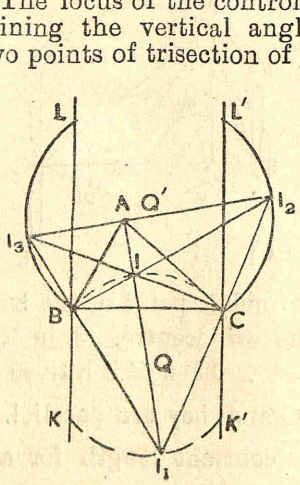
(Examine this line when the triangle is equilateral.)

Examples.

(1) *Given the base and the vertical angle of a triangle determine the loci of the in-centre, the ex-centres, the orthocentre, the nine-point centre and the centroid.*

(i) Given the base and the vertical angle, the circumcircle can be drawn and the position of the circumcentre is fixed. The locus of the controid is a segment of the circle containing the vertical angle on the base formed by the two points of trisection of the base. (P. 81.)

(ii)



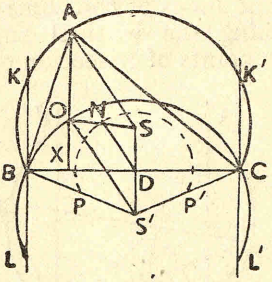
It has been shown (P. 77) that if Q be the middle point of the arc of the circum-circle ABC in which A does not lie, $QB = QI = QC = QI_1$.

$\therefore$ I and I_1 lie on the segments of the circle whose centre is the fixed point Q and whose radius is equal to QB.

$\because$ BI, CI bisect the base angles, $\angle BIC = 180^\circ - \frac{1}{2}B - \frac{1}{2}C = 90^\circ + \frac{1}{2}A$. Therefore I lies on the minor arc BC of the above circle. Similarly $\angle BI_1C = 90^\circ - \frac{1}{2}A$ and I_1 lies on the major arc BC of that circle; but since, I_1BC and I_1CB are both acute angles (being halves of the exterior angles at the base), only the portion of the major segment included between the perpendiculars to the base at B and C is the locus of I_1. The loci of I and I_1 are shown by discontinuous lines.

Again $\angle BI_2C = \angle BI_3C = \frac{1}{2}A$ and therefore I_2 and I_3 lie on the major arc of the circle on BC containing the angle $\frac{1}{2}A$. In this case-$\angle I_3BC = 90^\circ + \frac{1}{2}B$ and $\angle I_2CB = 90^\circ + \frac{1}{2}C$. Therefore the portion of the arc lying between the perpendiculars to the base at B and C is not a part of the locus. The loci of I_2 and I_3 are shown by continuous arcs in the figure.

(iii)



Let D be the middle point of the base, S the circum-centre and O the orthocentre. Join SD and produce it to S' so that DS' = SD. Then S' is a fixed point.

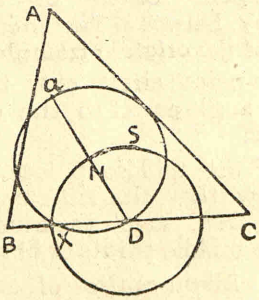
SS' = 2 SD = AO and they are parallel.

$\therefore$ S'O = AS = a constant length for all positions of A.

$\therefore$ O lies on the circle whose centre is S′ and whose radius is equal to the circum-radius AS. Since S′O is always parallel and equal to SA, the locus of O is a segment of the circle exactly corresponding to the segment BAC of the circum-circle. The arcs LB, BC and CL′ of the locus of O are the parts corresponding to the arcs BK, KK′ and K′C of the circum-circle. (K, K′ and L, L′ are the pairs of points in which the perpendiculars to the base at B and C meet the loci of A and O.)

(iv) The nine-point centre N is the middle point of OS. Therefore DN is parallel to and half of S′O or SA. The nine-point centre lies on the circle whose centre is the middle point D of BC and whose radius is half of the circum-radius, but the entire circle is not the locus of N. AO being equal to SS′ and parallel to it, N the mid-point of OS is also the mid-point of AS′, A can move from B to C on the arc BAC of the circum-circle and consequently N can lie anywhere between P and P′ which are the points of intersection of S′B and S′C with the circle whose centre is D and radius is half the circum-radius. This arc (PNP′) shown by a discontinuous line in the diagram is the locus of N. Obviously P and P′ are also the middle points of SL and SL′ respectively, since L and L′ are the extreme positions of the orthocentre.

(2) *Given the base and the vertical angle of a triangle show that the nine-point circle touches a fixed circle.*



When the base BC and the vertical angle A are given, the circum-circle can be drawn. Therefore the points B, C, S, and D the mid-point of BC, become fixed.

D is a point on the nine-point circle.

α, the mid-point of AO, is the point diametrically opposite to D on the nine-point circle whose centre is N and therefore α, N, D are collinear and $\alpha D = R$, $\alpha N = ND = \frac{1}{2} R$, R being the circum-radius.

$\therefore$ The circle whose centre is D and radius R touches the circle whose centre is N and radius $\frac{1}{2}$ R, at the point α. They have internal contact, the distance between their centres being equal to the difference between their radii.

$\therefore$ The nine-point circle always touches the fixed circle whose centre is D and whose radius is equal to the circum-radius of $\triangle$ ABC.

Exercise 12.

(1) If O is the ortho-centre of a triangle ABC, then the triangles OBC, OCA, OAB, ABC have the same nine-point circle.

(2) If a triangle ABC is right-angled at A, show that the nine-point circle touches the circum-circle at A. (*m*)

(3) **The nine-point centre of the middle point triangle is midway between the circum-centre and the nine-point centre of the original triangle.**

(4) The nine-point circle cuts the base BC of a triangle ABC at an angle equal to the difference between the angles B and C.

(5) If D, E, F are middle points of the sides of a triangle ABC, show that the nine-point circles of the triangles AEF, BFD, CDE, touch the nine-point circle of the triangle DEF at the middle points of EF, FD, DE respectively.

(6) S is the circum-centre of a triangle ABC and AS meets BC in T. Prove that the circle on AT as diameter touches the nine-point circle.

(7) Prove that $SO^2 = R^2 - 4Rr'$ where R is the circum-radius of $\triangle ABC$ and r' the in-radius of the pedal triangle.

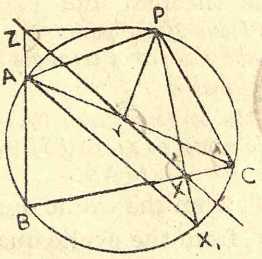
(8) If perpendiculars are drawn from the ortho-centre of a triangle ABC to the bisectors of the angle A, their feet are collinear with the middle point of BC and the nine-point centre.

(9) If α is the middle point of OA, and if αG meets the circum-circle at Q, prove that αD is parallel to AQ.

(10) Construct the triangle ABC in which $O\alpha = 1{\cdot}1$ cm., $OD = 3{\cdot}2$ cm., $\alpha D = 3{\cdot}5$ cm., α and D being the middle points of OA and BC respectively.

Theorem 37.

The feet of the perpendiculars drawn to the sides of a triangle from any point on the circum-circle of the triangle are collinear.



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Let ABC be a triangle and let the perpendiculars to the sides BC, CA, AB from any point P on the circle ABC meet the sides in X, Y, Z respectively. Join PA, PC, XY, YZ.

$\angle PYZ = \angle PAZ$, since P, Y, A, Z are concyclic.

and $\angle PAZ = \angle PCB$, since A, B, C, P are concyclic.

Again since $\angle PXC$, $\angle PYC$ are right angles, P, Y, X, C are concyclic.

$\therefore \angle PCB$ or $\angle PCX = 180° - \angle PYX$.

$\therefore \angle PYZ + \angle PYX = \angle PCB + \angle PYX = 180°$.

$\therefore$ **X, Y, Z lie on a straight line.**

Notes. (1) XYZ is called the **pedal line** or the **Simson line** of the point P.

The pedal lines of the vertices are the altitudes while those of the diametrically opposite points are the sides of the triangle.

(2) *If the feet of the perpendiculars from a point to the sides of any triangle are collinear, the point lies on the circum-circle of the triangle.*

This is the *converse* of the present theorem.

Proof. With the same diagram as before, let X, Y, Z, the feet of the perpendiculars from P to the sides, be collinear.

Then $\angle PYZ = \angle PCB$, since P, Y, X, C are concyclic and YZ is the produced part of XY.

Again $\angle PYZ = \angle PAZ$, $\because$ P, Z, A, Y are concyclic.

$\therefore \angle PAZ = \angle PCB$ and P lies on the circle ABC.

(3) The theorem and its converse give rise to the result: *The locus of the point, the feet of the perpendiculars from which to the sides of a triangle are collinear, is the circum-circle of the triangle.*

(4) *If the perpendicular* PX *to* BC *is produced to meet the circum-circle again at* X_1, *and if* PY *is the perpendicular to* CA, *then* XY *is parallel to* AX_1.

For, $\angle AX_1X = \angle ACP$, from the cyclic quadrilateral $CPAX_1$

$= \angle PXY$, from the cyclic quadrilateral PYXC.

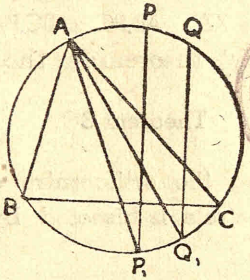
$\therefore AX_1$ is parallel to XY.

Similarly AX_1 is parallel to XZ, where PZ is the perpendicular from P to AB.

Since XY, XZ are both parallel to AX_1, X, Y, Z are collinear.

This provides us with *an alternative proof of the theorem* and also give us *a construction for drawing the Simson line of a point* P. We draw PXX_1 perpendicular to BC and meeting BC, at X and the circum-circle at X_1 and draw the Simson line XYZ parallel to AX_1 through X.

(5) *The pedal lines of two points* P *and* Q *on the circum-circle of a triangle* ABC *intersect at the same angle as that subtended by* PQ *at the circumference of the circle.*



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If the perpendiculars to BC from P, Q meet the circle again at P_1, Q_1, the pedal lines of P and Q are respectively parallel to AP_1 and AQ_1 and intersect each other at an angle equal to $\angle P_1AQ_1$, the angle subtended at the circumference by the arc P_1Q_1 or by the equal arc PQ.

As a particular case, *the pedal lines of diametrically opposite points on the circum-circle of a triangle intersect at right angles*, for if P, Q be diametrically opposite points, PQ subtends a right angle at the circumference.

Example.

If the pedal line of P *cuts the sides* BC, CA, AB *of a triangle* ABC *in* X, Y, Z, *show that*

$$2R.XY = AB.PC, \quad 2R.YZ = BC.PA, \quad 2R.ZX = CA.PB.$$

R *being the radius of the circum-circle of the triangle.*

Hence deduce **Ptolemy's theorem** *for a cyclic quadrilateral.*

Let P be a point on the arc AC. (Refer to the diagram of Th. 37.) Since PYC, PXC are right angles, PC is a diameter of the circle through P, Y, X, C. In circles CYX, CAB, the chords XY, AB subtend the same angle at C.

$\therefore XY : AB = PC : 2R.$ [See Ex. 6, (6)]

$\therefore 2R.XY = AB.PC.$

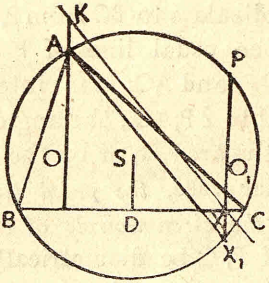
Similarly $2R.YZ = BC.PA$ and $2R.ZX = CA.PB$.

$\because XY + YZ = XZ$, $AB.PC + BC.PA = CA.PB$,

the result of Ptolemy's theorem for the quadrilateral PABC.

Theorem 38.

The line joining the orthocentre of a triangle to any point on the circum-circle, is bisected by the pedal line of the point.



Let O be the orthocentre of the triangle ABC and P any point on the circum-circle. Let S be the circum-centre and D the middle point of BC. Let the perpendicular from P to BC meet it in X and the circle in X_1. Join AX_1, and draw the Simson line of P parallel to AX_1 through X and let it meet OA in K. In XP take O_1 so that $XO_1 = X_1X$.

$\because XO_1 = X_1X$, O_1 is the orthocentre of $\triangle$ PBC. (Th. 33)

$\therefore PO_1 = 2SD = AO$.

Also $O_1X = XX_1 = AK$, since $AKXX_1$, having its opposite sides parallel, is a parallelogram,

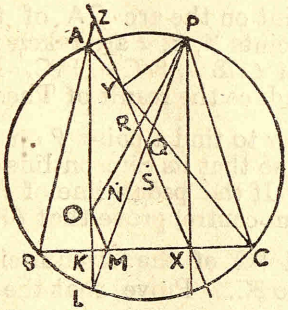
$\therefore PX = PO_1 + O_1X = OA + AK = OK$.

$\therefore$ PX is equal and parallel to OK.

$\therefore$ OX is equal and parallel to KP, and PKOX is a parallelogram.

$\therefore$ OP is bisected by KX, which is the pedal line of P.

Alternative proof.



Let AO produced meet the base at K and circum-circle at L. Let X, Y, Z be the feet of the perpendiculars from P to BC, CA, AB and hence XYZ the pedal line which cuts OP at R. Let PL cut BC at M and the pedal line at Q. Join PC.

$\angle QXM = 90° - \angle QXP = 90° - \angle YCP$, ($\because$ Y, X, C, P are concyclic)

$= 90° - \angle OLM$ on the same arc AP of $\bigcirc$ ABC,

$= \angle LMK = \angle QMX$

$= \angle OMK$, since $\triangle OMK = \triangle MLK$.

$\therefore$ QR is parallel to OM.

$\because \angle QXM = \angle QMX$ and $\angle QPX = \angle QXP$,

$QM = QX = QP$.

$\therefore$ QR, parallel to OM, cuts OP at R, its middle point.

$\therefore$ The pedal line bisects OP.

Note. Since R is the middle point of OP and N the middle point of OS, NR is equal to half SP or equal to the radius of the nine-point circle.

$\therefore$ R, the point where OP is bisected by the pedal line of P, lies on the nine-point circle.

The pedal line of P', the point on $\bigcirc$ ABC diametrically opposite to P, passes through R', the point diametrically opposite to R on the nine-point circle.

Exercise 13.

(1) P is a point on the arc CA of the circum-circle of △ ABC. The points X, Y, Z are taken in BC, CA, AB respectively such that ∠ PZB = ∠ PXC = ∠ PYC. Prove that X, Y, Z are collinear. Deduce the result of Theorem 37.

(2) Show how to find a point P on the circum-circle of a triangle ABC, so that its Simson line may be parallel to a given line. If the pedal line of P is parallel to AS, (S being the circum-centre) prove that AP is parallel to BC.

(3) A chord PQ of the circum-circle of a triangle ABC is parallel to BC. Prove that the pedal line of P is perpendicular to AQ. (*m*)

(4) *Prove that the circum-circles of the four triangles formed by four non-concurrent straight lines, have a common point (Wallace point). Show that the orthocentres of the four triangles are collinear.* (*m*)

(5) If the pedal line of a point on the circum-circle of a triangle is equally inclined to two sides, the point bisects the arc intercepted by the third side.

(6) If the perpendiculars from a point P on the circum-circle of a triangle ABC to the sides meet the circle again at L, M, N, prove that the triangle LMN is congruent with the triangle ABC.

(7) Show that the pedal lines of three points on the circum-circle of a triangle form a triangle, similar to that formed by the three points.

(8) ABC is a triangle and S is its circum-centre. SD, perpendicular to BC, meets the circum-circle at P. Prove that the line through D perpendicular to AP, will bisect OP, O being the orthocentre.

(9) *Show that the pedal lines of the extremeties of a diameter of the circum-circle intersect at right angles on the nine-point circle.* (*m*)

(10) If the pedal line of P cuts the sides BC, CA, AB of △ ABC in X, Y, Z so that XY = YZ, show that PA : PC = AB : BC.

CHAPTER IV.

SOLID GEOMETRY.

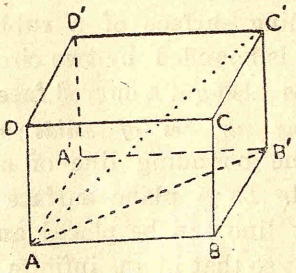
In addition to the two dimensions, the length and the breadth, if an object possesses a third dimension, the thickness, the depth or the height (according to the point of view) it is said to be a solid. The simplest example of this is a box.

Solids are bounded by surfaces, just as plane figures are bounded by lines and the lines themselves are bounded by points. For example, the box has six bounding surfaces or faces—the top and the bottom, the front and the back, and the left and the right sides. Again the bounding surface of a rubber ball is spherical. A cylinder is bounded by two circular flat faces at the ends and has also got a curved face. Hence we see that *the bounding surfaces of solids may be plane or curved*, just as the bounding line of a figure may be straight or curved. In a plane surface such as a black board, a straight line can be placed and turned round about a point on it so that in an infinite number of positions every point of the line lies on the surface. Some times it is possible to place a straight line on a curved surface so that every point of it lies on the surface, as in the case of the straight lines joining the vertex of a cone to the points on the circular base or of the straight lines parallel to the axis, on a cylinderical surface; but if such straight lines are very slightly displaced, they cease to lie entirely on that surface. The following property may be taken to define a plane: The straight line joining *any* two points on a plane lies entirely in the plane.

Surfaces intersect along lines, straight or curved.

We may conceive of a line as a closely packed collection of points or as we say, a line is generated by a point. Similarly a surface may be generated by lines and a solid may be generated by surfaces. For example, when the hypotenuse of a right-angled triangle rotates about a side, it generates a conical surface.

In solid geometry we deal with points and lines not necessarily coplanar or lying in a plane. The following definitions and explanations will be found to be useful. Any three-dimensional object cannot be fully represented on a two-dimensional paper and the student is advised to note carefully the method of drawing figures to represent solids.



The above figure represents a box, the dotted lines indicating the edges which cannot be seen from the present position.

ABCD, A′B′C′D′ form the front and the back faces,
ABB′A′, DCC′D′ form the bottom and the top faces,
AA′D′D, BB′C′C form the left and right faces.

This solid (supposed to be an ordinary box, but called the *cuboid* or the *rectangular solid* or the *rectangular parallelopiped)* has rectangles for all its faces. The pairs of rectangles mentioned above are situated in parallel or non-intersecting planes. If we consider the planes containing two rectangles in any other

combination, they will be found to have a common line of intersection, called an *edge* of the solid.

Generalising the result we may say that **two given planes may be parallel, in which case they have no common point, or they may intersect, in which case the points of intersection lie on a straight line.**

Again there are twelve edges or bounding lines for this solid. Of these there are three sets of four parallel edges (AB, A′B′, CD, C′D′), (BC, B′C′, AD, A′D′), (AA′, BB′, CC′, DD′). Any two edges of any set are parallel and have the same direction because they lie in the same plane and do not meet.

The edge AB is parallel to three edges A′B′, CD, C′D′ and meets four edges AA′, AD, BB′, BC. The remaining four edges CC′, DD′, A′D′, B′C′ neither intersect AB, nor are they parallel to it. In fact we cannot imagine a plane in which both the lines CC′ and AB can lie. Such lines are called *skew lines.*

We may generalise the results thus—**Any two straight lines may (i) intersect in a point or (ii) be parallel, *i.e.*, coplanar and non-intersecting or (iii) be skew to each other, *i.e.*, non-coplanar.**

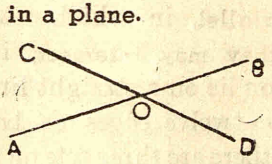
Next consider the relationship between the edges and the surfaces of the solid. Four lines (AB, BE′, B′A′, AA′) lie in the plan ABB′A′, four lines (AD, BC, B′C′, A′D′) intersect that plane, each in a point and four lines (DC, CC′, C′D′, DD′) are parallel to the same plane, *i.e.*, the plane and the lines have no common point.

In general, any straight line may lie wholly within a given plane, or intersect it in a point, or not meet it at all, in which case, it is said to be parallel to the plane.

The following are some of the useful theorems in Solid Geometry.

Theorem 39.

Any two straight lines which intersect each other, or are parallel, lie in a plane.



Let AB and CD be two straight lines intersecting at O. Imagine a plane containing the line AB and revolving about it (as the pages of a book about the common edge). In its various positions we get different planes, all containing AB. In one position the plane will pass through the point D of the line CD. In that position of the revolving plane, it contains D and O. By the property of a plane, the entire line COD lies in that plane.

If AB, CD are parallel, by definition they are coplanar.

Notes. (1) This method gives us a construction for obtaining a plane containing two intersecting lines.

(2) If AB and CD are parallel or intersecting straight lines intersected by EF at P and Q, then the three lines are coplanar, for P and Q lie in the plane determined by AB and CD and hence the entire line PQ or EF lies in it.

$\therefore$ AB, CD, EF are coplanar.

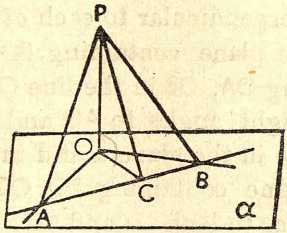
Any triange ABC *lies entirely in one plane*; but the sides of a quadrilateral need not be coplanar, as is easily seen by slightly folding a quadrilateral drawn on paper, about a diagonal. Such quadrilaterals are called *skew quadrilaterals*.

(3) Two straight lines cannot lie in the same plane, only when they are *skew* to each other.

(4) *A plane is uniquely determined by* (i) *three non-collinear points,* (ii) *a straight line and a point lying outside it,* or (iii) *two parallel or intersecting straight lines.*

Theorem 40.

If a straight line is perpendicular to each of two intersecting straight lines at their points of intersection, it is perpendicular to any line passing through that point and lying in the plane determined by the two intersecting lines.



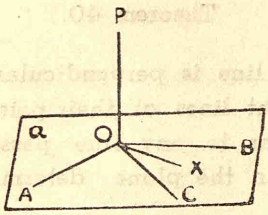
Let OP be a straight line meeting the lines OA and OB at O at right angles. Let OC be any other line in the plane α, determined by OA and OB. Through any point C on that line, draw a straight line terminated by OA, OB at A, B, so as to be bisected at C. (In case OC lies within the angle supplement to ∠AOB, draw the line bisected at C and terminated by OA at A and by BO produced at B'.) The proof applies to that case after replacing B by B'.) Join PA, PB, PC,

PC and OC bisect AB and by Apollonius' theorem.

$$AP^2 + BP^2 = 2\,PC^2 + 2\,AC^2,$$
$$AO^2 + BO^2 = 2\,OC^2 + 2\,AC^2,$$
$$\therefore (AP^2 - AO^2) + (BP^2 - BO^2) = 2\,(PC^2 - OC^2).$$
$$\because \angle AOP = \angle BOP = 90°,$$
$$AP^2 - AO^2 = OP^2 \quad \text{and} \quad BP^2 - BO^2 = OP^2.$$
$$\therefore OP^2 + OP^2 = 2\,(PC^2 - OC^2) \quad \text{or} \quad OP^2 = PC^2 - OC^2.$$
$$\therefore \angle POC \text{ is a right angle.}$$

Notes. (1) *If a straight line is perpendicular to each of three concurrent straight lines at their point of intersection, the three straight lines are coplanar.*

(This is the *converse* of the theorem.)



Let PO be perpendicular to each of the lines OA, OB, and OC. Let the plane containing PO, OC intersect the plane α containing OA, OB in the line OX.

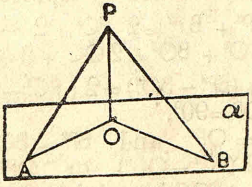
∵ PO is at right angles to AO and OB, it is perpendicular to every line in the plane α and in particular to OX.

∴ In the plane containing PO, OC, the straight line OP is perpendicular to both OX and OC.

∴ OX coincides with OC and OA, OB, OC are coplanar.

(2) A line, which meets a plane so that it cuts at right angles every line in the plane which it meets, is called a perpendicular or normal to the plane. It is easy to see that there is only one such perpendicular to a plane at any point in it. All others are oblique lines.

(3) Of all the lines that can be drawn from a point P to a given plane, the perpendicular is the shortest and those obliques are equal, which cut the plane at points equidistant from the foot of the perpendicular.

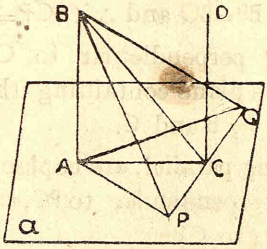


If PQ be the perpendicular to a plane AOB, in the triangle POA right-angled at O, the hypotenuse PA is greater than PO.

Again, if PA and PB be two obliques so that OA = OB, then △ POA is equal in all respects to △ POB and PA = PB.

Theorem 41.

Two straight lines which are perpendicular to the same plane are parallel.



Let AB, CD be two straight lines perpendicular to a plane α and let them meet it in A and C. Join AC. In the plane α draw the perpendicular to AC through C and take any two points P and Q on it equidistant from C. Join AP, AQ, BP, BC, BQ.

In the triangle APC and ACQ.

AC is common, $PC = CQ$, $\angle ACP = \angle ACQ = 90°$.

$\therefore$ The triangles are equal in all respects and $AP = AQ$.

In the triangles ABP and ABQ,

AB is common, $AP = AQ$. and $\angle BAP = \angle BAQ$ ($\because$ AB is $\perp$ to α).

$\therefore$ The triangles are congruent and $BP = BQ$.

Again the triangles BPC and BCQ, which have the sides of the one equal to those of the other, are equal in all respects.

$$\therefore \angle BCP = \angle BCQ = 90°.$$

$\therefore$ CP is perpendicular to CA, CB, CD, which are hence coplanar lines. That is, CD lies in the plane determined by CA and CB, which also contains AB, since A and B, the points on CA and CB, lie in it.

$\therefore$ AB and CD are coplanar and $\angle BAC = 90° = \angle ACD$.

$\therefore$ AB and CD are parallel.

Notes. (1) The *converse* is also true: **If AB and CD are parallel lines and AB is perpendicular to any plane, CD is also perpendicular to the same plane.**

With the same diagram, we can prove that
AP = AQ and ∴ BP = BQ and ∴ ∠ BCP = ∠ BCQ = 90°.

Now CP being perpendicular to CA and CB, it is perpendicular to the plane containing them. The plane contains the points A, B and C.

AB and CD being parallel, are coplanar. Since A, B, C lie in one plane perpendicular to PC, DC also lies in it and is perpendicular to CP.

DC is perpendicular to AC, since AB, CD are parallel. ∴ DC is perpendicular to the plane α containing AC, PC.

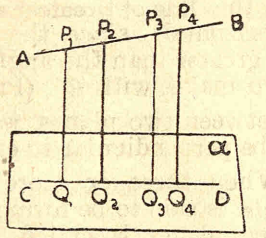
(2) If AB be perpendicular to a plane α which it meets at A and if AC be the perpendicular from A to any line PQ in the plane meeting it at C, then BC is perpendicular to PQ. This result is called **the theorem of three perpendiculars.**

(3) This theorem gives us *a method for drawing a line perpendicular to a plane from an external point.* For if B be a point outside a plane α, draw BC perpendicular to any line PQ in α and meeting it in C. In the plane α draw the line perpendicular to PQ through C and draw BA perpendicular to that line so as to meet it at A. BA is the required perpendicular to the plane α.

(4) **Two straight lines, each of which is parallel to a third straight line, are themselves parallel.** For let l, m be two lines parallel to a third line n and let α be a plane perpendicular to n. Then each of the lines l, m, which are parallel to n, are perpendicular to α and are therefore parallel to each other.

Theorem 42.

The projection of a straight line on a plane is a straight line.



If from the various points P_1, P_2, P_3, . . . of a straight line AB, perpendiculars P_1Q_1, P_2Q_2, P_3Q_3. . . are drawn to a plane α, they are all coplanar; for, P_1Q_1 and P_2Q_2 are both perpendicular to α and are therefore parallel; P_2Q_2 lies in the plane containing P_1Q_1 and AB, and similarly P_3Q_3, . . . all lie in the same plane. [Th. 39, Note 2.]

The feet of the perpendiculars Q_1, Q_2, Q_3, . . . are points on the straight line CD of intersection of that plane containing the perpendiculars, with the plane α.

∴ The projection of AB is the straight line CD.

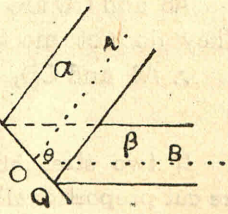
Notes. (1) **The projection of a straight line on a plane parallel to it, is a straight line parallel to the given line;** for if AB does not meet the plane α it does not meet the line CD, and they are coplanar.

(2) **The angle between a line and a plane is defined as the angle between the line and its projection on the plane.** It is the least of the angles which the line makes with the lines it meets in the plane.

(*Prove this result*)

(3) **The angle between two planes is called a dihedral angle.** It is measured thus:

Let the two planes α, β meet in the line PQ. Through any point O in PQ draw OA, OB perpendicular to PQ in the planes α, β respectively. ∠AOB is the dihedral angle. It is also the angle which OA makes with the plane β or which OB makes with α.



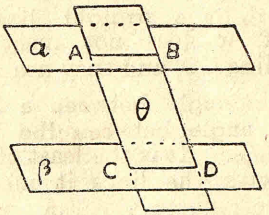
OA is called the line of greatest slope on the plane α (when β is horizontal), since the angle which OA makes with β is greater than the angle which any other line through O in α makes with β, (Prove the result).

If the angle between two planes is a right angle, the planes are said to be perpendicular to each other.

(4) When three or more planes meet at a point, a solid angle is said to be formed. The solid angle contained by three planes is called a trihedral angle. To measure a solid angle, describe a sphere with the point of concurrence of the planes as centre. The ratio of the area of the spherical surface cut off by the solid angle to the entire surface area of the sphere is a measure of the solid angle. The elements of the solid angle are the faces meeting at the point, the edges which are the intersections of consecutive faces, the face angles between consecutive edges and the dihedral angles between consecutive plane faces.

Theorem 43.

If two parallel planes are cut by a third plane, the lines of intersection are parallel.



Let the plane θ cut the parallel planes α and β in the straight line AB and CD.

AB and CD are coplanar, since they lie in θ.

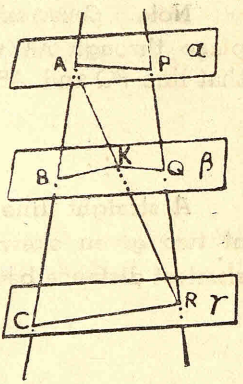
They do not meet, since they lie in parallel planes α, β.

$\therefore$ AB and CD, the lines of intersection, are parallel.

Theorem 44.

If two straight lines are cut by parallel planes they are cut proportionally.

Let α, β, γ be three parallel planes cutting two given lines at (A, B, C) and (P, Q, R) respectively.



Let AR meet the plane β at K.

Join BK, CR, KQ, AP.

(Note that B, K, Q need not lie on a straight line.)

BK and CR are parallel, for they are the lines of intersection of the parallel planes β and γ with the plane ACR.

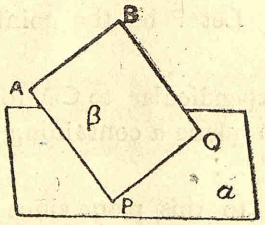
KQ and AP are parallel, for they are the lines of intersection of the parallel planes α and β with the plane RAP.

$$\therefore AB : BC = AK : KR = PQ : QR.$$

$\therefore$ AC and PR are cut proportionally by the parallel planes.

Theorem 45.

If a straight line is parallel to any straight line in a given plane, it is parallel to the plane.



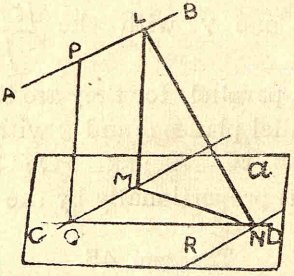
If AB is parallel to the line PQ in the plane α, then AB and PQ lie in the same plane β which intersects α along the the line PQ. Since AB does not meet PQ, it has no point in common with α, for all the points common to β and α lie on PQ.

$\therefore$ AB is parallel to the plane α.

Note. *Conversely*, if AB is parallel to a plane α, then any plane through AB will cut α in a line parallel to AB, for that line PQ and AB are coplanar and non-intersecting.

Theorem 46.

A straight line can be drawn perpendicular to both of two given skew lines and this straight line is the shortest distance between the lines.



Let AB and CD be two given skew lines. Draw RN, any line parallel to AB, to cut CD at N. Draw the projection QM of AB on the plane α containing CD and RN, to cut CD at Q. Let P be the point of which Q is the projection.

(1) PQ is perpendicular to CD, for PQ is drawn perpendicular to the plane α containing CD and RN ($\because$ Q is the projection of P).

AB is parallel to this plane since it is parallel to RN in the plane.

$\therefore$ AB is parallel to its projection QM. [Th. 42., Note 1.]

$\because$ PQ is perpendicular to QM, it is also perpendicular to AB which is parallel to QM.

$\therefore$ PQ is the common perpendicular to the skew lines AB and CD.

(ii) If the projection of any other point L of AB on the plane α is M, then LM is less than the oblique LN joining L to any point N in CD.

PQML being a parallelogram, PQ = LM.

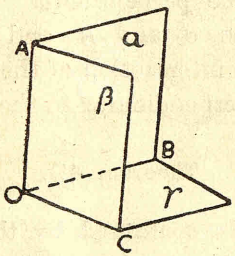
$\therefore$ PQ < LN.

$\therefore$ PQ is the shortest distance.

Note. *The angle between two skew lines* CD *and* AB *is defined as the angle between* CD *and* QM *which is parallel to* AB *and which cuts* CD.

Theorem 47.

If a straight line is perpendicular to a plane, then any plane containing the line is also perpendicular to the given plane.



Let OA be the straight line perpendicular to the plane γ, and α any plane containing OA and intersecting γ along the line OB. Draw OC in the plane γ, perpendicular to OB.

$\because$ OA is perpendicular to γ, $\angle AOC = 90° = \angle AOB$.

$\because$ OA and OC are both perpendicular to OB, $\angle AOC$ measures the dihedral angle between the planes α and γ.

$\therefore$ The plane α is perpendicular to the plane γ.

Note. *Conversely*, if the planes α and γ are perpendicular to each other, any line OA drawn in the plane α perpendicular to the line of intersection, is perpendicular to γ.

It may be further proved that the perdendicular from any point A in the plane α, to the perpendicular plane γ, lies in the plane α.

Theorem 48.

If two intersecting planes are each perpendicular to the third plane, their line of intersection is also perpendicular to that plane.

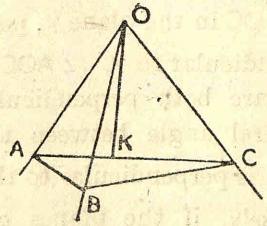
(Refer to the diagram of Theorem 47.)

Let α and β be two intersecting planes each perpendicular to the third plane γ. The perpendicular from any point A in α to the plane γ lies in α. If A be a point on the common line of intersection of the planes α and β, the perpendicular from A to γ lies in each of the planes α and β and thefore coincides with AO the line of intersection of the planes α and β.

$\therefore$ AO is perpendicular to the plane γ.

Theorem 49.

In a solid angle contained by three planes meeting at a point, any two plane angles are together greater than the third.



Let a solid angle be formed at O by the intersections of three planes OAB, OBC and OCA, its plane angles being $\angle$AOB, $\angle$BOC and $\angle$COA.

It is required to show that the sum of any two of these angles is greater than the third. It is *sufficient* to prove that the sum of the two smaller angles is greater than the greatest angle, say $\angle AOC$ in the figure.

From any point A in OA draw AB, AC equally inclined to OA in the planes OAB, OCA respectively, so that they intersect the edges OB and OC in B and C respectively. Join BC. In the plane AOC draw the line OK making $\angle AOK = \angle AOB$ and intersecting AC in K.

In the triangles AOB and AOK,

AO is common, $\angle OAB = \angle OAK$, $\angle AOB = \angle AOK$.

$\therefore$ The triangles are congruent and

$$OB = OK \quad \text{and} \quad AB = AK.$$

In the triangle ABC, $AB + BC > AC$,

$$\text{or} \quad AB + BC > AK + KC.$$

$$\text{Since } AB = AK \qquad BC > KC.$$

In the triangles OBC and OKC.

OC is common, $OB = OK$ and $BC > KC$.

$\therefore \angle BOC$ opposite to the greater side is greater than $\angle KOC$ opposite to the less.

Adding to these the equal angles AOB and AOK,

$\angle BOC + \angle AOB > \angle KOC + \angle AOK$ or $> \angle AOC$.

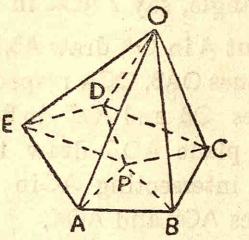
$\therefore$ Any two plane angles are together greater than the third.

Note. The analogous theorem in plane geometry is the following:

Any two sides of a triangle are together greater than the third. In corresponding theorems of plane geometry and solid geometry, we often find the following pairs of magnitudes, analogues—the side and the plane angle, the angle and the dihedral angle, the triangular area and the trihedral angle.

Theorem 50.

The plane angles at the vertex of a convex solid angle are together less than four right angles.



Let O be the vertex of a solid angle at which the plane angles AOB, BOC, COD, ... meet. Let any plane intersect the solid angle in the plane rectilineal figure ABCD ... Let any point P in that plane be joined to the vertices A, B, C, ... of the figure.

Corresponding to each triangle in the surface such as AOB, there is a triangle such as APB in the plane base. Therefore the sum of all the angles of the triangles meeting at O is equal to the sum of all the angles of the triangles into which ABCD ... is divided.

At the trihedral angles at the vertices A, B, C, ... the sum of any two face angles is greater than the third.

$\therefore \angle ABO + \angle OBC > \angle ABC$,

$\angle BCO + \angle OCD > \angle BCD$,

$\angle CDO + \angle ODE > \angle CDE$, and so on.

Adding them all, we find that the sum of the base angles of the triangles meeting at O is greater than the sum of the angles of the rectilineal figure in the base.

The sum of the angles of the triangles meeting at O

= the sum of the angles of the triangles in the base

= the sum of the angles of the rectilineal figure in the base + the angles at P.

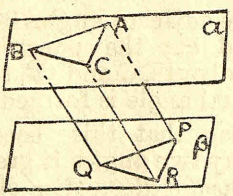
$\therefore$ The sum of the plane angles at O is less than the sum of the angles at P, that is, less than four right angles.

Note. It can be proved that *there are only five regular polyhedra* bounded by regular figures of the same kind. They are *the tetrahedron, the hexahedron (cube), the octahedron, the dodecahedron and the icosahedron* consisting respectively 4, 6, 8, 12, 20 faces. These are also called **Platonic solids.**

Euler has shown that in any convex *polyhedron or* a solid bounded by planes, $F + V = E + 2$, where F, V, E are the numbers of the faces, the vertices and the edges of the polyhedron. (Verify the result in the case of the simple solids—the cube, the prism or the pyramid.)

Examples.

(1) *If two straight lines in a plane are respectively parallel to two other straight lines in another plane, the angle between the first pair of lines is equal to the angle between the second pair.*



Let ABC be an angle in the plane α such that its arms AB and BC are respectively parallel to PQ and QR, the arms of the angle PQR in the plane β.

Take the points A, C, P, R, so that BA = QP and BC = QR.
Then, since AB and PQ are equal and parallel,
AP and BQ are equal and parallel.
Similarly BC being equal and parallel to QR,
CR and BQ are equal and parallel.
$\therefore$ AP and CR are equal and parallel.
$\therefore$ AC and PR are equal and parallel.
$\therefore$ $\triangle^s$ ABC and PQR are congruent and $\angle$ ABC = $\angle$ PQR.

(2) *The lengths of the concurrent edges of a rectangular box are* 2·1, 1·6, 1·2 *feet. Calculate the length of a diagonal and the angles it makes with the largest face and the longest edge.*

The diagram on page 104 represents the box in which AB = 2·1 ft., AA′ = 1·6 ft., AD = 1·2 ft.

All the faces are rectangles.

$\because \angle ABB' = 90°$, $AB'^2 = AB^2 + BB'^2 = 2{\cdot}1^2 + 1{\cdot}6^2$.

B′C′ is perpendicular to the face ABB′A′ and $\angle AB'C' = 90°$.

$\therefore AC'^2 = AB'^2 + B'C'^2 = (2{\cdot}1^2 + 1{\cdot}6^2) + 1{\cdot}2^2$

$= 4{\cdot}41 + 2{\cdot}56 + 1{\cdot}44 = 8{\cdot}41$. $\therefore AC' = 2{\cdot}9$ ft.

AB′ is the projection of AC′ on the plane ABB′A′, since C′B′ is perpendicular to that face. The angle between AC′ and that face is $C'AB' = \theta$, say.

Then $\sin\theta = C'B'/AC' = 1{\cdot}2/2{\cdot}9 = {\cdot}4138$. $\therefore \theta = 24° 27'$.

The angle $C'AB = \alpha$, between AC′ and the edge AB, is given by $\cos\alpha = AB/AC' = 2{\cdot}1/2{\cdot}9 = {\cdot}7241$. $\therefore \alpha = 43° 36'$.

Exercise 14.

(1) In the diagram of Theorem 40, if ACB represents any line drawn perpendicular to OC so as to cut OA and OB at A and B, prove that PC is perpendicular to AB and hence deduce an *alternative proof* of the theorem.

(2) Draw three adjacent angles BOC, COA, AOB′ on a sheet of paper and cut the paper along OB and OB′. Fold the diagram about OC and OA so that OB and OB′ coincide. A trihedral angle is formed at O.

It is easily seen that this becomes possible only when the sum of any two angles is greater than the third.

This is an *experimental illustration* of Theorem 49.

(3) ABCD is a rectangle in which AB = 8″ and BC = 6″. Through the centre O of the rectangle, a straight line OP equal to 12″ is drawn perpendicular to the plane of the rectangle. Calculate PA and the angle it makes with the plane ABCD.

(4) P is a point outside the plane of a triangle ABC and equidistant from its vertices. If the perpendicular from P to the plane cuts it at S, show that S is the circumcentre of the triangle ABC. (*m*)

Hence deduce *a construction for drawing a perpendicular to any plane from an external point.*

If AB=7·2″, BC=7·8″, CA=3·0″, SP=5·2″, find PA.

(5) Through the circum-centre S of an equilateral triangle ABC whose sides are each 6″ long, a line SP, 2″ long, is drawn perpendicular to the plane ABC. Find PA, ∠PAS and the dihedral angle between the planes PAB and ABC.

(6) ABCD is a skew quadrilateral. Show that the mid-points of AB, BD, DC, AC form a parallelogram whose plane is parallel to AD or BC.

(7) A, B, C, D are the vertices of a skew qurdrilateral. Show that the lines joining the mid points of the pairs of lines (AB, CD), (BC, AD), (AC, BD) bisect one another.

(8) **The angle between the perpendiculars from any point to two intersecting planes is equal to the dihedral angle between them.**

(9) PQ is the line of greatest slope on a plane α inclined at 30° to a horizontal plane β, their line of intersection being QR. If QS is drawn on the inclined plane making 60° with QP, find the angles which it makes with QR. and with the horizontal plane.

(10) Show that the sum of the angles of a skew quadrilateral is less than four right angles.

(11) OA, OB, OC are three concurrent edges of a rectangular box. If P be the vertex diagonally opposite to O, and if OP makes angles α, β, γ with OA, OB, OC, show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(12) OA, OB, OC are three concurrent lines, each of which is perpendicular to the other two. If OP is perpendicular to the plane ABC, show that P is the orthocentre of the triangle ABC. *(m)*

(13) The perpendicular from P to the plane of a triangle ABC meets it in the orthocentre of the triangle. Show that if one of the three angles APC, CPB, BPA is a right angle, the others are also right angles. *(m)*

(14) AB is the shortest distance between two skew lines and CD is any other line meeting them at C and D. Show that the line joining the mid-points of AB and CD is perpendicular to AB.

(15) A model of a cuboid is made of wire, showing its edges and one of its diagonals. If the lengths of the three edges meeting at any corner be 12″, 9″ and 8″ respectively, find the length of the wire used.

(16) ABCD is a face of a cube and O the centre of the opposite face. Find the angle between OA and the face ABCD and also the angle between the planes OAB and ABCD.

(17) A rectangular solid stands on a base of 12·0 sq. inches. Its height is 3·6 in., and its diagonal is 6·5 in. Find its length and breadth and calculate the area of its whole surface.

(18) Show that the sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the straight lines joining the mid-points of its opposite edges.

(19) ABCD is a regular tetrahedron, the length of each edge being 6″. Find the surface area of the solid and the length of the perpendicular from D to the opposite face. If θ be the angle between the faces ABC and ABD, prove that $\sin^2 \theta = 8/9$.

(20) If p be the perpendicular from any vertex of a regular tetrahedron to the opposite face and if each edge is equal to $2a$, show that $3p^2 = 8a^2$.

(21) Show that the angle between two adjacent faces of a regular tetrahedron is $\cos^{-1}(\frac{1}{3})$. (*m*)

(22) If opposite pairs of edges of a tetrahedron are equal show that the sum of the face angles at each vertex is two right angles and that each face angle is acute. (*m*)

(23) Prove that the shortest distance between two opposite edges of a regular tetrahedron is one half of the diagonal of the square on an edge. (*m*)

(24) If OA, OB and OC are the edges of a rectangular solid, meeting at O and are of lengths a, b and c respectively, prove that the area of the triangle ABC is $\frac{1}{2}\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$ and the length of the perpendicular from O to the plane ABC is $abc / \sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$.

ANSWERS

NUMERICAL EXAMPLES

EXERCISE 5.

(1) 3.0″, 1.8″. (2) 4.5″, 1.8″. (3) 1·56 cm., 3.64 cm., 3.9 cm., 9·1 cm. (4) (i) 1·6 cm., 4·0 cm., (ii) 2·67 cm., (iii) 2·25 cm., 3.375 sq. cm., (iv) 1 : 16, (5) 4.0″, 3.5″, 25 : 16.

EXERCISE 6.

(15) 30 ft., 4 ft.

EXERCISE 7.

(1) (i) 6.0 cm., 7.5 cm., (ii) 4.05 cm., 6.75 cm., (iii) 3.6 cm., 4.8 cm., (iv) 16 : 81. (2) 2.0 cm., 2.25 cm., 16 : 25. (3) 7 : 24.

EXERCISE 9.

(3) 2 : 1, 4 : 1, 4 : 9. (4) 124.8 yds., 6336 sq. yds. (5) 18,000 sq. miles. (6) 12.15 acres.

EXERCISE 14.

(3) 13″, 67° 23′. (4) 6.5″. (5) 4″, 30°, 49° 6′. (9) 30°, 14° 29′. (15) 133″. (16) 54° 44′, 63° 26′. (17) 4.8″, 2.5″, 76.56 sq. in. (19) $36\sqrt{3}$ (=62 35) sq.in., $2\sqrt{6}$ (=4.9) in.

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